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GENERAL PRODUCT FORMULA OF MULTIPLE INTEGRALS OF LÉVY PROCESS

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Abstract. We derive a product formula for finitely many multiple stochastic integrals of Lévy process, expressed in terms of the associated Poisson random measure. The formula is compact. The proof is short and uses the exponential vectors and polarization techniques.

1. Introduction

Stochastic analysis of nonlinear functionals of Lévy processes (including Brownian motion and Poisson process) have been studied extensively and found many applications. There have been already many standard books on this topic [1, 7, 8]. In the analysis of nonlinear Wiener (Brownian) functional the Wiener-Itô chaos expansion to expand a nonlinear functional of Brownian motion into the sum of multiple Wiener-Itô integrals is a fundamental contribution to the field. The product formula to express the product of two (or more) multiple integrals as linear combinations of some other multiple integrals is one of the important tools ([2]). It plays an important role in stochastic analysis, e.g. Malliavin calculus ([2, 6]).

The product formula for two multiple integrals of Brownian motion is known since the work of [9, Section 4] and the general product formula can be found for instance in [2, chapter 5]. In this paper we give a general formula for the product of m multiple integrals of the Poisson random measure associated with (purely jump) Lévy process. The formula is in a compact form and it reduced to the Shigekawa's formula when $m = 2$ and when the Lévy process is reduced to Brownian motion.

When $m = 2$ a similar formula was obtained in [3], where the multiple integrals is with respect to the Lévy process itself (ours is with respect to the associated Poisson random measure which has better properties). To obtain their formula in [3] Lee and Shih use white noise analysis framework. In this work, we only use the classical framework in hope that this work is accessible to a different spectrum of readers.

The product formula for multiple Wiener-Itô integrals of the Brownian motion plays an important role in many applications such as in U-statistics [4]. We hope similar things may happen. But we are not pursuing this goal in the current paper. Our formula is for purely jump Lévy process. It can be combined with the

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classical formulas [2, 4, 6, 9] so that it holds for general Lévy process (including the continuous component).

This paper is organized as follows. In Section 2, we recall some preliminaries on Lévy process, the associated Poisson random measure, multiple integrals. We also state our main result in this section. In Section 3, we give the proof of the formula.

2. Preliminary and Main Results

Let $T > 0$ be a positive number and let $\{\eta(t) = \eta(t, \omega), 0 \le t \le T\}$ be a Lévy process on some probability space $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_t, 0 \le t \le T\}$ satisfying the usual condition. This means that $\{\eta(t)\}$ has independent and stationary increment and the sample path is right continuous with left limit. Without loss of generality, we assume $\eta(0) = 0$. If the process $\eta(t)$ has all moments for any time index t, then presumably, one can use the polynomials of the process to approximate any nonlinear functional of the process $\{\eta(t), 0 \le t \le T\}$. However, it is more convenient to use the associated Poisson random measure to carry out the stochastic analysis of these nonlinear functionals.

The jump of the process η at time t is defined by

$$\Delta\eta(t) := \eta(t) - \eta(t-) \quad \text{if } \Delta\eta(t) \neq 0\,.$$

Denote $\mathbb{R}_0 := \mathbb{R}\backslash\{0\}$ and let $\mathcal{B}(\mathbb{R}_0)$ be the Borel σ-algebra generated by the family of all Borel subsets $U \subset \mathbb{R}$, such that $\bar{U} \subset \mathbb{R}_0$. If $U \in \mathcal{B}(\mathbb{R}_0)$ with $\bar{U} \subset \mathbb{R}_0$ and $t > 0$, we then define the Poisson random measure $N : [0, T] \times \mathcal{B}(\mathbb{R}_0) \times \Omega \to \mathbb{R}$, associated with the Lévy process η by

$$N(t, U) := \sum_{0 \le s \le t} \chi_U(\Delta\eta(s))\,, \tag{2.1}$$

where χ_U is the indicator function of U. The associated Lévy measure ν of η is defined by

$$\nu(U) := \mathbb{E}[N(1, U)] \tag{2.2}$$

and the compensated jump measure $\tilde{N}$ is defined by

$$\tilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt\,. \tag{2.3}$$

The stochastic integral $\int_{\mathbb{T}} f(s, z)\tilde{N}(ds, dz)$ is well-defined for a predictable process $f(s, z)$ such that $\int_{\mathbb{T}} \mathbb{E}|f(s, z)|^2\nu(dz)ds < \infty$, where and throughout this paper we use $\mathbb{T}$ to represent the domain $[0, T] \times \mathbb{R}_0$ to simplify notation.

Let

$$\hat{L}^{2,n} := \left(L^2(\mathbb{T}, \lambda \times \nu)\right)^{\otimes n} \subseteq L^2\left(\mathbb{T}^n, (\lambda \times \nu)^n\right)$$

be the space of symmetric, deterministic real functions f such that

$$\|f\|^2_{\hat{L}^{2,n}} = \int_{\mathbb{T}^n} f^2(t_1, z_1, \cdots, t_n, z_n)dt_1\nu(dz_1)\cdots dt_n\nu(dz_n) < \infty\,,$$

where $\lambda(dt) = dt$ is the Lebesgue measure. In the above the symmetry means that

$$\begin{aligned}(t_1, z_1, \cdots, t_i, z_i, \cdots, t_j, z_j, \cdots, t_n, z_n)\\ = f(t_1, z_1, \cdots, t_j, z_j, \cdots, t_i, z_i, \cdots, t_n, z_n)\end{aligned}$$

for all $1 \le i < j \le n$. For any $f \in \hat{L}^{2,n}$ the multiple Wiener-Itô integral

$$I_n(f) := \int_{\mathbb{T}^n} f(t_1, z_1, \cdots, t_n, z_n) \tilde{N}(dt_1, dz_1) \cdots \tilde{N}(dt_n, dz_n) \tag{2.4}$$

is well-defined. The importance of the introduction of the associated Poisson measure and the multiple Wiener-Itô integrals are in the following theorem which means that any square integrable nonlinear functional F of the Lévy process η can be expanded as sum of multiple Wiener-Itô integrals.

Theorem 2.1 (Wiener-Itô chaos expansion for Lévy process). *Let $\mathcal{F}_T = \sigma(\eta(t), 0 \le t \le T)$ be the σ-algebra generated by the Lévy process η.*

Let $F \in L^2(\Omega, \mathcal{F}_T, P)$ be an $\mathcal{F}_T$ measurable square integrable random variable. Then F admits the following chaos expansion:

$$F = \sum_{n=0}^{\infty} I_n(f_n), \tag{2.5}$$

where $f_n \in \hat{L}^{2,n}, n = 1, 2, \cdots$ and where we denote $I_0(f_0) := f_0 = \mathbb{E}(F)$. Moreover, we have

$$\|F\|^2_{L^2(P)} = \sum_{n=0}^{\infty} n!\|f_n\|^2_{\hat{L}^{2,n}}. \tag{2.6}$$

This chaos expansion theorem is one of the fundamental results in stochastic analysis of Lévy processes. It has been widely studied in particular when η is the Brownian motion (Wiener process). We refer to [2], [6], [7] and references therein for further reading.

To state our main result of this paper, we need some notation. Fix a positive integer $m \ge 2$. Denote

$$\Upsilon = \Upsilon_m = \{\mathbf{i} = (i_1, \cdots, i_\alpha),\ \alpha = 2, \cdots, m,\ 1 \le i_1 < \cdots < i_\alpha \le m\} \tag{2.7}$$

where $\alpha = |\mathbf{i}|$ is the length of the multi-index $\mathbf{i}$ (we shall use α, β to denote a natural number). It is easy to see that the cardinality of Υ is $\kappa_m := 2^m - 1 - m$. Denote $\vec{\mathbf{i}} = (\mathbf{i}_1, \cdots, \mathbf{i}_{\kappa_m})$, which is an unordered list of the elements of Υ, where $\mathbf{i}_\beta \in \Upsilon$. We use $\vec{l} = (l_{\mathbf{i}_1}, \cdots, l_{\mathbf{i}_{\kappa_m}})$ to denote a multi-index of length κ_m associated with Υ, where $l_{\mathbf{i}_\alpha}$, $1 \le \alpha \le \kappa_m$ are nonnegative integers. $\vec{l}$ can be regarded as a function from Υ to $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$. Denote

$$\begin{cases} \Omega = \left\{\vec{l}, \vec{n} : \Upsilon \to \mathbb{Z}_+\right\} \qquad \text{and for any } \vec{l},\, \vec{n} \in \Omega\,, \\ \chi(k, \vec{l}, \vec{n}) = \sum_{1 \le \alpha \le \kappa_m} \left[l_{\mathbf{i}_\alpha} \chi_{\{\mathbf{i}_\alpha \text{ contains } k\}} + n_{\mathbf{i}_\alpha} \chi_{\{\mathbf{i}_\alpha \text{ contains } k\}} \right]. \end{cases} \tag{2.8}$$

The above χ on the right hand side refers to the indicate function. Denote $\chi(\vec{l}, \vec{n}) = (\chi(1, \vec{l}, \vec{n}), \cdots, \chi(m, \vec{l}, \vec{n}))$. The conventional notations such as $|\vec{l}| = l_{\mathbf{i}_1} + \cdots + l_{\mathbf{i}_{\kappa_m}}$; $\vec{l}! = l_{\mathbf{i}_1}! \cdots l_{\mathbf{i}_{\kappa_m}}!$ and so on are in use. Notice that we use $l_{\mathbf{i}_1}$ instead of l_1 to emphasize that the $l_{\mathbf{i}_1}$ corresponds to $\mathbf{i}_1$. For $\mathbf{i} = (i_1, \cdots, i_\alpha), \mathbf{j} = (j_1, \cdots, j_\beta) \in \Upsilon$, and non negative integers μ and ν denote

$$\begin{aligned}\hat{\otimes}_{\mathbf{i}}^{\mu}(f_1,\cdots,f_m) &= \int_{([0,T]\times\mathbb{R}_0)^{\mu}} f_{i_1}((s_1,z_1),\cdots,(s_\mu,z_\mu),\cdots)\hat{\otimes}\cdots \\ &\qquad \hat{\otimes} f_{i_\alpha}((s_1,z_1),\cdots,(s_\mu,z_\mu),\cdots)ds_1\nu(dz_1)\cdots \\ &\qquad ds_\mu\nu(dz_\mu)\ f_1\hat{\otimes}\cdots\hat{\otimes}\hat{f}_{i_1}\hat{\otimes}\cdots\hat{\otimes}\hat{f}_{i_\alpha}\cdots\hat{\otimes}f_m\,,\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}V_{\mathbf{j}}^{\nu}(f_1,\cdots,f_m) &= f_{j_1}((s_1,z_1),\cdots,(s_\nu,z_\nu),\cdots)\hat{\otimes}\cdots \\ &\quad \hat{\otimes} f_{j_\beta}((s_1,z_1),\cdots,(s_\nu,z_\nu),\cdots)f_1\hat{\otimes}\cdots\hat{\otimes}\hat{f}_{j_1}\hat{\otimes}\cdots\hat{\otimes}\hat{f}_{j_\beta}\cdots\hat{\otimes}f_m\,,\end{aligned} \tag{2.10}$$

where $\hat{\otimes}$ denotes the symmetric tensor product and $\hat{f}_{j_1}$ means that the function f_{j_1} is removed from the list. Let us emphasize that both $\hat{\otimes}_{\mathbf{i}}^{\mu}$ and $V_{\mathbf{j}}^{\nu}$ are well-defined when the lengths of $\mathbf{i}$ and $\mathbf{j}$ are one. However, we shall not use $\hat{\otimes}_{\mathbf{i}}^{\mu}$ when $|\mathbf{i}|=1$ and when $|\mathbf{j}|=1$, $V_{\mathbf{j}}^{\nu}(f_1,\cdots,f_m)=f_1\hat{\otimes}\cdots\hat{\otimes}f_m$ (namely, the identity operator). For any two elements $\vec{l}=(l_{\mathbf{i}_1},\cdots,l_{\mathbf{i}_{\kappa_m}})$ and $\vec{n}=(\mu_{\mathbf{j}_1},\cdots,\mu_{\mathbf{j}_{\kappa_m}})$ in Ω, denote

$$\begin{aligned}\hat{\otimes}_{\vec{\mathbf{i}}}^{\vec{l}} &= \hat{\otimes}_{\mathbf{i}_1,\cdots,\mathbf{i}_{\kappa_m}}^{l_{\mathbf{i}_1},\cdots,l_{\mathbf{i}_{\kappa_m}}} = \hat{\otimes}_{\mathbf{i}_1}^{l_{\mathbf{i}_1}}\cdots\hat{\otimes}_{\mathbf{i}_{\kappa_m}}^{l_{\mathbf{i}_{\kappa_m}}}\,, \\ V_{\vec{\mathbf{j}}}^{\vec{n}} &= V_{\mathbf{j}_1,\cdots,\mathbf{j}_{\kappa_m}}^{\mu_{\mathbf{j}_1},\cdots,\mu_{j_{\kappa_m}}} = V_{\mathbf{j}_1}^{\mu_{\mathbf{j}_1}}\hat{\otimes}\cdots\hat{\otimes}V_{\mathbf{j}_{\kappa_m}}^{\mu_{\mathbf{j}_{\kappa_m}}}\,.\end{aligned} \tag{2.11}$$

Now we can state the main result of the paper.

Theorem 2.2. *Let $q_1,\cdots,q_m$ be positive integers greater than or equal to 1. Let*

$$f_k\in\left(L^2([0,T]\times\mathbb{R}_0,dt\otimes\nu(dz))\right)^{\hat{\otimes}q_k}\,,\quad k=1,\cdots,m\,.$$

Then

$$\begin{aligned}\prod_{k=1}^m I_{q_k}(f_k) = &\sum_{\substack{\vec{l},\vec{n}\in\Omega \\ \chi(1,\vec{l},\vec{n})\le q_1 \\ \cdots \\ \chi(m,\vec{l},\vec{n})\le q_m}} \frac{\prod_{k=1}^m q_k!}{\prod_{\alpha=1}^{\kappa_m} l_{\mathbf{i}_\alpha}!\prod_{\beta=1}^{\kappa_m}\mu_{\mathbf{j}_\beta}!\prod_{k=1}^m(q_k-\chi(k,\vec{l},\vec{n}))!} \\ &I_{|q|+|\vec{n}|-|\chi(\vec{l},\vec{n})|}\big(\hat{\otimes}_{\mathbf{i}_1,\cdots,\mathbf{i}_{\kappa_m}}^{l_{\mathbf{i}_1},\cdots,l_{\mathbf{i}_{\kappa_m}}}\hat{\otimes}V_{\mathbf{j}_1,\cdots,\mathbf{j}_{\kappa_m}}^{\mu_{\mathbf{j}_1},\cdots,\mu_{j_{\kappa_m}}}(f_1,\cdots,f_m)\big)\,,\end{aligned} \tag{2.12}$$

where we recall

$$|q|=q_1+\cdots+q_m \quad\text{and}\quad |\chi(\vec{l},\vec{n})|=\chi(1,\vec{l},\vec{n})+\cdots+\chi(m,\vec{l},\vec{n})\,.$$

Remark 2.3. The above formula looks sophisticated and it may be understood in the following manner. There are two types of contraction operation involved in the above formula. The first one is the *integration contraction*: we choose certain subset of functions $f_{i_1},\cdots,f_{i_\alpha}$ and we choose μ variables (throughout the paper for simplicity we call a pair (s,z) as one variable) in each of these chosen functions and set them to be the same: $(s_1,z_1),\cdots,(s_\mu,z_\mu)$ and we integrate with respect to these variables (with respect to the product measure of $ds\nu(dz)$) as in (2.9). The second one is the *simple contraction* without integration: we also choose certain subset of functions $f_{j_1}, \cdots, f_{j_\beta}$ and let ν variables in all of these functions be the same: $(s_1, z_1), \cdots, (s_\nu, z_\nu)$, as in (2.10). We just concatenate the remaining variables: The concatenation of function $g_1(x_1, \cdots, x_{n_1})$, $\cdots$, $g_m(x_1, \cdots, x_{n_m})$ means

$$g_1(x_{1,1}, \cdots, x_{1,n_1}) \cdots g_m(x_{m,1}, \cdots, x_{m,n_m}) .$$

All the variables not integrated out with respect to $ds\nu(dz)$ will be integrated with respect to the Poisson random measure. The summation in the formula (2.12) is over all the possible two contraction operations. See the following examples 2.5-2.6 for more explanation.

Remark 2.4. If the index $\vec{n}$ does not appear, then there will be no operator V. In this case the formula (2.12) becomes [2, formula 5.3.5], which is the product formula for finitely many multiple integrals of Brownian motion.

Example 2.5. If $m = 2$, then $\kappa_m = 2^2 - 1 - 2 = 1$. To shorten the notations we can write $q_1 = n$, $q_2 = m$, $f_1 = f_n$, $f_2 = g_m$, $l_{\alpha_1} = l$, $n_{\beta_1} = k$. Thus, $\chi(1, \vec{l}, \vec{n}) = \chi(2, \vec{l}, \vec{n}) = l + k$ and $|q| + |\vec{n}| - |\chi(\vec{l}, \vec{n})| = n + m + k - 2(l + k) = n + m - 2l - k$. Hence the formula (2.12) becomes the following. If

$$f_n \in \left(L^2([0,T] \times \mathbb{R}_0, dt \otimes \nu(dz))\right)^{\hat{\otimes} n}$$

and

$$g_m \in \left(L^2([0,T] \times \mathbb{R}_0, dt \otimes \nu(dz))\right)^{\hat{\otimes} m} ,$$

then

$$I_n(f_n) I_m(g_m) \quad = \sum_{\substack{k,l \in \mathbb{Z}_+ \\ k+l \le m \wedge n}} \frac{n!m!}{l!k!(n-k-l)!(m-k-l)!} I_{n+m-2l-k}\left(f_n \hat{\otimes}_{k,l} g_m\right) ,$$

where $\mathbb{Z}_+$ denotes the set of non negative integers and

$$\begin{aligned} & f_n \hat{\otimes}_{k,l} g_m(s_1, z_1, \cdots, s_{n+m-k-2l}, z_{n+m-k-2l}) \\ & = \text{symmetrization of } \int_{\mathbb{T}^l} f_n(s_1, z_1, \cdots, s_{n-l}, z_{n-l}, t_1, y_1, \cdots, t_l, y_l) \\ & \qquad g_m(s_1, z_1, \cdots, s_k, z_k, s_{n-l+1}, \cdots, z_{n-l+1}, \cdots, \\ & \qquad s_{n+m-k-2l}, z_{n+m-k-2l}, t_1, z_1, \cdots, t_l, z_l) dt_1 \nu(dz_1) \cdots dt_l \nu(dz_l). \end{aligned} \tag{2.13}$$

Example 2.6. If $m = 3$, then $\kappa_m = 2^3 - 1 - 3 = 4$. The set

$$\Upsilon_3 = \{\mathbf{i}_1 = (1,2), \mathbf{i}_2 = (2,3), \mathbf{i}_3 = (1,3), \mathbf{i}_4 = (1,2,3)\}$$

We also write

$$\begin{aligned} & l_{\mathbf{i}_1} = l_{12} \,,\ l_{\mathbf{i}_2} = l_{23} \,,\ l_{\mathbf{i}_3} = l_{13} \,,\ l_{\mathbf{i}_4} = l_{123} \,, \\ & \mu_{\mathbf{j}_1} = k_{12} \,, \mu_{\mathbf{j}_2} = k_{23} \,, \mu_{\mathbf{j}_3} = k_{13} \,, \mu_{\mathbf{j}_4} = k_{123} \,. \end{aligned}$$

Thus,

$$\begin{aligned} \chi(1, \vec{l}, \vec{n}) &= l_{12} + l_{13} + l_{123} + k_{12} + k_{13} + k_{123} \,, \\ \chi(2, \vec{l}, \vec{n}) &= l_{12} + l_{23} + l_{123} + k_{12} + k_{23} + k_{123} \,, \\ \chi(3, \vec{l}, \vec{n}) &= l_{13} + l_{23} + l_{123} + k_{13} + k_{23} + k_{123} \,, \end{aligned}$$

and

$$|q|+|\vec{n}|-|\chi(\vec{l},\vec{n})| = q_1+q_2+q_3-2l_{12}-2l_{23}-2l_{13}-3l_{123} -k_{12}-k_{23}-k_{13}-2k_{123}\,.$$

Hence we have

$$I_{q_1}(f_1)I_{q_2}(f_2)I_{q_3}(f_3) \tag{2.14}$$

$$= \sum_{\substack{l_{ij},k_{ij}\geq 0\\ \chi(i,\vec{l},\vec{n})\leq q_i\;,i=1,2,3}} \frac{q_1!q_2!q_3!}{l_{12}!l_{13}!l_{23}!l_{123}!k_{12}!k_{13}!k_{23}!k_{123}!\prod_{r=1}^{3}(q_i-\chi(i,\vec{l},\vec{n}))!}$$

$$\cdot I_{|q|+|\vec{n}|-|\chi(\vec{l},\vec{n})|}\left(\hat{\otimes}_{\vec{\mathbf{i}}}^{\vec{l}}\,\hat{\otimes}V_{\vec{\mathbf{j}}}^{\vec{n}}(f_1,f_2,f_3)\right). \tag{2.15}$$

The above contraction operator $\hat{\otimes}_{\vec{\mathbf{i}}}^{\vec{l}}\,\hat{\otimes}V_{\vec{\mathbf{j}}}^{\vec{n}}$ is given as follows:

$$\hat{\otimes}_{\vec{\mathbf{i}}}^{\vec{l}}\,\hat{\otimes}V_{\vec{\mathbf{j}}}^{\vec{n}}(f_1,f_2,f_3)(\vec{s_1},\vec{s_2},\vec{s_3},\vec{s}_{12},\vec{s}_{13},\vec{s}_{23},\vec{s}_{123})$$

$$= \text{symmetrization of } \int_{\mathbb{T}^{|l|}} f_1(\vec{r}_{12},\vec{r}_{13},\vec{r}_{123},\vec{s}_{12},\vec{s}_{13},\vec{s}_{123},\vec{s_1})$$

$$f_2(\vec{r}_{12},\vec{r}_{23},\vec{r}_{123},\vec{s}_{12},\vec{s}_{23},\vec{s}_{123},\vec{s_2})f_3(\vec{r}_{13},\vec{r}_{23},\vec{r}_{123},\vec{s}_{13},\vec{s}_{23},\vec{s}_{123},\vec{s_3})$$

$$\nu(d\vec{r}_{12})\nu(d\vec{r}_{13})\nu(d\vec{r}_{23})\nu(d\vec{r}_{123})\,, \tag{2.16}$$

where (denoting $|l|=l_{12}+l_{13}+l_{23}+l_{123}$ and $|k|=k_{12}+k_{13}+k_{23}+k_{123}$)

$$\vec{r}_{12} = ((s_1,z_1),\cdots,(s_{l_{12}},z_{l_{12}}))\,,$$
$$\vec{r}_{13} = ((s_{l_{12}+1},z_{l_{12}+1}),\cdots,(s_{l_{12}+l_{13}},z_{l_{12}+l_{13}}))\,,$$
$$\vec{r}_{23} = ((s_{l_{12}+l_{13}+1},z_{l_{12}+l_{13}+1}),\cdots,(s_{l_{12}+l_{13}+l_{23}},z_{l_{12}+l_{13}+l_{23}}))\,,$$
$$\vec{r}_{123} = ((s_{l_{12}+l_{13}+l_{23}+1},z_{l_{12}+l_{13}+l_{23}+1}),\cdots,(s_{|l|},z_{|l|}))\,,$$
$$\vec{s}_{12} = ((s_{|l|+1},s_{|l|+1}),\cdots,(s_{|l|+k_{12}},z_{|l|+k_{12}}))\,,$$
$$\vec{s}_{13} = ((s_{|l|+k_{12}+1},z_{|l|+k_{12}+1}),\cdots,(s_{|l|+k_{12}+k_{13}},z_{|l|+k_{12}+k_{13}}))\,,$$
$$\vec{s}_{23} = ((s_{|l|+k_{12}+k_{13}+1},z_{|l|+k_{12}+k_{13}+1}),\cdots,$$
$$(s_{|l|+k_{12}+k_{13}+k_{23}},z_{|l|+k_{12}+k_{13}+k_{23}}))\,,$$
$$\vec{s}_{123} = ((s_{|l|+k_{12}+k_{13}+k_{23}+1},z_{|l|+k_{12}+k_{13}+k_{23}+1}),\cdots,(s_{|l|+|k|},z_{|l|+|k|}))\,;$$

for $i=1,2,3$, $\vec{s_i}$ represents the remaining variables in f_i and there are $q_i-\chi(i,\vec{l},\vec{n})$ variables (we count every pair (s,z) as one variable) in $\vec{s}_i$. In (2.16), the variables marked as $\vec{r}$ are integrated out. The total number of variables appeared in all $\vec{s}$ is

$$|q|+|\vec{n}|-|\chi(\vec{l},\vec{n})| = q_1+q_2+q_3-2l_{12}-2l_{23}-2l_{13}-3l_{123} -k_{12}-k_{23}-k_{13}-2k_{123}$$

and they will be integrated with respect to the Poisson random measure as a multiple integral.

Remark 2.7. When η is the Brownian motion, the product formula (2.13) is known since [9] (see e.g. [2, Theorem 5.6] for a formula of the general form (2.13)) and is given by

$$I_n(f_n)I_m(g_m) = \sum_{l=0}^{n\wedge m} \frac{n!m!}{l!(n-l)!(m-l)!} I_{n+m-2l}\Big(f_n \hat{\otimes}_l g_m\Big)\,. \tag{2.17}$$

It is a "special case" of (2.12) when $k=0$.

3. Proof of Theorem 2.2

We shall prove the main result (Theorem 2.2) of this paper. We shall prove this by using the polarization technique (see [2, Section 5.2]). First, let us find the Wiener-Itô chaos expansion for the *exponential functional* (random variable) of the form

$$\begin{aligned} Y(T) &= \mathcal{E}(\rho(s,z)) \\ &:= \exp\left\{\int_{\mathbb{T}} \rho(s,z)\tilde{N}(dz,ds) - \int_{\mathbb{T}} \Big(e^{\rho(s,z)} - 1 - \rho(s,z)\Big)\nu(dz)ds\right\} \end{aligned} \tag{3.1}$$

where $\rho(s,z) \in \hat{L}^2 := \hat{L}^{2,1} = L^2(\mathbb{T}, \nu(dz)\otimes\lambda(dt))$. An application of Itô formula (see e.g. [7]) yields

$$Y(T) = 1 + \int_0^T \int_{\mathbb{R}_0} Y(s-)\Big[\exp\left(\rho(s,z)\right) - 1\Big]\tilde{N}(ds,dz)\,.$$

Repeatedly using this formula, we obtain the chaos expansion of $Y(T)$ as follows.

$$\begin{aligned} \mathcal{E}(\rho(s,z)) &= \exp\left\{\int_{\mathbb{T}} \rho(s,z)\tilde{N}(dz,ds) - \int_{\mathbb{T}} \Big(e^{\rho(s,z)} - 1 - \rho(s,z)\Big)\nu(dz)ds\right\} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} I_n(f_n)\,, \end{aligned} \tag{3.2}$$

where the convergence is in $L^2(\Omega, \mathcal{F}_T, P)$ and

$$f_n = f_n(s_1, z_1, \cdots, s_n, z_n) = (e^{\rho} - 1)^{\hat{\otimes} n} = \prod_{i=1}^n \left(e^{\rho(s_i,z_i)} - 1\right). \tag{3.3}$$

We shall first make critical application of the above expansion formula (3.2)-(3.3). For any functions $p_k(s,z) \in \hat{L}^2$ (in what follows when we write k we always mean $k = 1, 2, \cdots, m$ and we shall omit $k = 1, 2, \cdots, m$), we denote

$$\rho_k(u_k, s, z) = \log(1 + u_k p_k(s,z))\,, \tag{3.4}$$

From (3.2)-(3.3), we have (consider u_k as fixed real numbers)

$$\mathcal{E}(\rho_k(u_k, s, z)) = \sum_{n=0}^{\infty} \frac{1}{n!} u_k^n I_n(f_{k,n})\,, \tag{3.5}$$

where

$$f_{k,n} = \frac{1}{u_k^n}\prod_{i=0}^n (e^{\rho_k(u_k,s_i,z_i)}-1) = p_k^{\otimes n} = \prod_{i=1}^n p_k(s_i,z_i) \tag{3.6}$$

It is clear that

$$\prod_{k=1}^m \mathcal{E}(\rho_k(u_k,s,z)) = \sum_{q_1,\cdots,q_m=0}^{\infty} \frac{1}{q_1!\cdots q_m!} u_1^{q_1}\cdots u_m^{q_m} I_{q_1}(f_{1,q_1})\cdots I_{q_m}(f_{m,q_m}) \tag{3.7}$$

where f_{k,q_k}, $k=1,\cdots,m$ are defined by (3.6). On the other hand, from the definition of the exponential functional (3.1), we have

$$\begin{aligned}
&\prod_{k=1}^m \mathcal{E}(\rho_k(u_k,s,z)) \\
&= \prod_{k=1}^m \exp\Big\{\int_{\mathbb{T}} \rho_k(u_k,s,z)\tilde{N}(dz,ds)\Big\} \qquad (3.8)\\
&\exp\Big\{-\int_{\mathbb{T}}\Big(e^{\rho_k(u_k,s,z)}-1-\rho_k(u_k,s,z)\Big)\nu(dz)ds\Big\}\\
&= \exp\Big\{\int_{\mathbb{T}}\sum_{k=1}^m \rho_k(u_k,s,z)\tilde{N}(dz,ds)\\
&-\int_{\mathbb{T}}\Big(e^{\sum_{k=1}^m \rho_k(u_k,s,z)}-1-\sum_{k=1}^m\rho_k(u_k,s,z)\Big)\nu(dz)ds\Big\}\\
&\cdot\exp\Big\{\int_{\mathbb{T}} e^{\sum_{k=1}^m \rho_k(u_k,s,z)}-\sum_{k=1}^m e^{\rho_k(u_k,s,z)}+m-1\Big)\nu(dz)ds\Big\}\\
&=: A\cdot B \qquad (3.9)
\end{aligned}$$

where A and B denote the above first and second exponential factors.

The first exponential factor A is an exponential functional of the form (3.1). Thus, again by the chaos expansion formula (3.2)-(3.3), we have

$$A = \sum_{n=0}^{\infty}\frac{1}{n!} I_n(h_n(u_1,\cdots,u_m))\,, \tag{3.10}$$

where

$$h_n(u_1,\cdots,u_m) = \prod_{i=0}^n (e^{\sum_{k=1}^m \rho_k(u_k,s_i,z_i)}-1)\,. \tag{3.11}$$

By the definition of q_k, we have

$$\sum_{k=1}^m \rho_k(u_k,s_i,z_i) \quad = \quad \log\prod_{k=1}^m (1+u_k p_k(s_i,z_i))\,.$$

Or

$$\begin{aligned} h_n(u_1, \cdots, u_m) &= \left(\left[\prod_{k=1}^m (1 + u_k p_k) - 1 \right] \right)^{\hat{\otimes} n} \\ &= \mathrm{Sym}_{(s_1,z_1),\cdots,(s_n,z_n)} \prod_{i=1}^n \left[\prod_{k=1}^m (1 + u_k p_k(s_i, z_i)) - 1 \right], \end{aligned}$$

where $\hat{\otimes}$ denotes the symmetric tensor product and $\mathrm{Sym}_{(s_1,z_1),\cdots,(s_n,z_n)}$ denotes the symmetriization with respect to $(s_1, z_1), \cdots, (s_n, z_n)$. Define

$$S = \{\mathbf{j} = (j_1, \cdots, j_\beta),\ \beta = 1, \cdots, m,\ 1 \le j_1 < \cdots < j_\beta \le m\} .$$

The cardinality of S is $|S| = \tilde{\kappa}_m := 2^m - 1$. We shall freely use the notations introduced in Section 2. Denote also

$$u_{\mathbf{j}} = u_{j_1} \cdots u_{j_\beta} , \quad p_{\mathbf{j}}(s,z) = p_{j_1}(s,z) \cdots p_{j_\beta}(s,z) \quad (\text{for } \mathbf{j} = (j_1, \cdots, j_\beta) \in S) .$$

We have

$$\begin{aligned} h_n(u_1, \cdots, u_m) = \left(\sum_{\mathbf{j} \in S} u_{\mathbf{j}} p_{\mathbf{j}} \right)^{\hat{\otimes} n} &= \sum_{|\vec{\mu}|=n} \frac{|\vec{\mu}|!}{\vec{\mu}!} u_{\vec{\mathbf{j}}}^{\vec{\mu}} p_{\vec{\mathbf{j}}}^{\hat{\otimes}\vec{\mu}} \\ = \sum_{\mu_{\mathbf{j}_1}+\cdots+\mu_{\mathbf{j}_{\tilde{\kappa}_m}}=n} & \frac{n!}{\mu_{\mathbf{j}_1}! \cdots \mu_{\mathbf{j}_{\tilde{\kappa}_m}}!} u_{\mathbf{j}_1}^{\mu_{\mathbf{j}_1}} \cdots u_{\mathbf{j}_{\tilde{\kappa}_m}}^{\mu_{\mathbf{j}_{\tilde{\kappa}_m}}} p_{\mathbf{j}_1}^{\hat{\otimes}\mu_{\mathbf{j}_1}} \hat{\otimes} \cdots \hat{\otimes} p_{\mathbf{j}_{\tilde{\kappa}_m}}^{\hat{\otimes}\mu_{\mathbf{j}_{\tilde{\kappa}_m}}} , \end{aligned}$$

where $\vec{\mu} : S \to \mathbb{Z}_+$ is a multi-index and we used the notation

$$u_{\vec{\mathbf{j}}}^{\vec{\mu}} = u_{\mathbf{j}_1}^{\mu_{\mathbf{j}_1}} \cdots u_{\mathbf{j}_{\tilde{\kappa}_m}}^{\mu_{\mathbf{j}_{\tilde{\kappa}_m}}} ; \qquad p_{\vec{\mathbf{j}}}^{\hat{\otimes}\vec{\mu}} = p_{\mathbf{j}_1}^{\hat{\otimes}\mu_{\mathbf{j}_1}} \hat{\otimes} \cdots \hat{\otimes} p_{\mathbf{j}_{\tilde{\kappa}_m}}^{\hat{\otimes}\mu_{\mathbf{j}_{\tilde{\kappa}_m}}} .$$

Inserting the above expression into (3.10) we have

$$A = \sum_{n=0}^\infty \sum_{\mu_{\mathbf{j}_1}+\cdots+\mu_{\mathbf{j}_{\tilde{\kappa}_m}}=n} \frac{1}{\mu_{\mathbf{j}_1}! \cdots \mu_{\mathbf{j}_{\tilde{\kappa}_m}}!} u_{\mathbf{j}_1}^{\mu_{\mathbf{j}_1}} \cdots u_{\mathbf{j}_{\tilde{\kappa}_m}}^{\mu_{\mathbf{j}_{\tilde{\kappa}_m}}} I_n(p_{\mathbf{j}_1}^{\hat{\otimes}\mu_{\mathbf{j}_1}} \hat{\otimes} \cdots \hat{\otimes} p_{\mathbf{j}_{\tilde{\kappa}_m}}^{\hat{\otimes}\mu_{\mathbf{j}_{\tilde{\kappa}_m}}}) \tag{3.12}$$

Now we consider the second exponential factor in (3.9):

$$\begin{aligned} B &= \exp\Big\{ \int_{\mathbb{T}} \Big(e^{\sum_{k=1}^m \rho_k(u_k,s,z)} - \sum_{k=1}^m e^{\rho_k(u_k,s,z)} + m - 1 \Big) \nu(dz) ds \Big\} \\ &= \exp\Big\{ \sum_{\mathbf{i} \in \Upsilon} u_{\mathbf{i}} \int_{\mathbb{T}} p_{\mathbf{i}}(s,z) \nu(dz) ds \Big\} , \end{aligned}$$

where Υ is defined by (2.7) (which is a subset of S such that $|\mathbf{j}| \geq 2$). Thus,

$$\begin{aligned} B &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{\mathbf{i}\in\Upsilon} u_{\mathbf{i}} \int_{\mathbb{T}} p_{\mathbf{i}}(s,z)\nu(dz)ds \right)^n \\ &= \sum_{n=0}^{\infty} \sum_{l_{\mathbf{i}_1}+\cdots+l_{\mathbf{i}_{\kappa_m}}=n} \frac{1}{l_{\mathbf{i}_1}!\cdots l_{\mathbf{i}_{\kappa_m}}!} u_{\mathbf{i}_1}^{l_{\mathbf{i}_1}} \cdots u_{\mathbf{i}_{\kappa_m}}^{l_{\mathbf{i}_{\kappa_m}}} \left(\int_{\mathbb{T}} p_{\mathbf{i}_1}(s,z)\nu(dz)ds \right)^{l_{\mathbf{i}_1}} \\ &\qquad \cdots \left(\int_{\mathbb{T}} p_{\mathbf{i}_{\kappa_m}}(s,z)\nu(dz)ds \right)^{l_{\mathbf{i}_{\kappa_m}}} , \end{aligned} \tag{3.13}$$

where $\vec{l} \in \Omega$ is a multi-index. Combining (3.12)-(3.13), we have

$$\begin{aligned} AB = \sum_{n,\tilde{n}=0}^{\infty} \sum_{\substack{\mu_{\mathbf{j}_1}+\cdots+\mu_{\mathbf{j}_{\tilde{\kappa}_m}}=n \\ l_{\mathbf{i}_1}+\cdots+l_{\mathbf{i}_{\kappa_m}}\tilde{n}}} & \frac{1}{\mu_{\mathbf{j}_1}!\cdots \mu_{\mathbf{j}_{\tilde{\kappa}_m}}! l_{\mathbf{i}_1}!\cdots l_{\mathbf{i}_{\kappa_m}}!} u_{\mathbf{j}_1}^{\mu_{\mathbf{j}_1}} \cdots u_{\mathbf{j}_{\tilde{\kappa}_m}}^{\mu_{\mathbf{j}_{\tilde{\kappa}_m}}} \\ & u_{\mathbf{i}_1}^{l_{\mathbf{i}_1}} \cdots u_{\mathbf{i}_{\kappa_m}}^{l_{l\mathbf{i}_{\kappa_m}}} B_{\mathbf{i},\mathbf{j},l_{\mathbf{i}},\mu_{\mathbf{j}}} , \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} B_{\mathbf{i},\mathbf{j},l_{\mathbf{i}},\mu_{\mathbf{j}}} := & \left(\int_{\mathbb{T}} p_{\mathbf{i}_1}(s,z)\nu(dz)ds \right)^{l_{\mathbf{i}_1}} \cdots \\ & \left(\int_{\mathbb{T}} p_{\mathbf{i}_{\kappa_m}}(s,z)\nu(dz)ds \right)^{l_{\mathbf{i}_{\kappa_m}}} I_n(p_{\mathbf{j}_1}^{\hat{\otimes}\mu_{\mathbf{j}_1}} \hat{\otimes} \cdots \hat{\otimes} p_{\mathbf{j}_{\tilde{\kappa}_m}}^{\hat{\otimes}\mu_{\mathbf{j}_{\tilde{\kappa}_m}}}) . \end{aligned} \tag{3.15}$$

To get an expression for $B_{\mathbf{i},\mathbf{j},l_{\mathbf{i}},\mu_{\mathbf{j}}}$ we use the notations (2.9)-(2.10) and (2.11). Then

$$B_{\mathbf{j},\tilde{\mathbf{j}},n_{\mathbf{j}},\tilde{n}_{\mathbf{j}}} = I_n(\hat{\otimes}_{\vec{\mathbf{i}}}^{\vec{l}} \hat{\otimes} V_{\vec{\mathbf{j}}}^{\vec{\mu}} (p_1^{\otimes n_{\mathbf{i}_1}}, \cdots, p_m^{\otimes n_m})) . \tag{3.16}$$

To compare the coefficients of $u_1^{n_{\mathbf{i}_1}} \cdots u_m^{n_m}$, we need to express the right hand side of (3.14) as a power series of $u_1, \cdots, u_m$. For $k = 1, \cdots, m$ denote

$$\tilde{\chi}(k, \vec{l}, \vec{\mu}) = \sum_{1\leq\alpha\leq\kappa_m} l_{\mathbf{i}_\alpha} I_{\{\mathbf{i}_\alpha \text{ contains } k\}} + \sum_{1\leq\beta\leq\tilde{\kappa}_m} \mu_{\mathbf{j}_\beta} I_{\{\mathbf{j}_\beta \text{ contains } k\}} . \tag{3.17}$$

Combining (3.9), (3.14) and (3.16), we have

$$\begin{aligned} & \sum_{q_1,\cdots,q_m=0}^{\infty} \frac{u_1^{q_1}\cdots u_m^{q_m}}{q_1!\cdots q_m!} I_{q_1}(p_1^{\otimes q_1}) \cdots I_{q_m}(p_m^{\otimes q_m}) \\ &= \sum_{n,\tilde{n}=0}^{\infty} \sum_{\substack{\mu_{\mathbf{j}_1}+\cdots+\mu_{\mathbf{j}_{\tilde{\kappa}_m}}=n \\ l_{\mathbf{i}_1}+\cdots+l_{\kappa_m}=\tilde{n} \\ \tilde{\chi}(k,\vec{l},\vec{\mu})=q_k, k=1,\ldots,m}} \frac{u_1^{q_1}\cdots u_m^{q_m}}{l_{\mathbf{i}_1}!\cdots l_{\mathbf{i}_{\kappa_m}}! \mu_{\mathbf{j}_1}!\cdots \mu_{\mathbf{j}_{\tilde{\kappa}_m}}!} \\ & \qquad I_n(\hat{\otimes}_{\mathbf{i}_1,\cdots,\mathbf{i}_{\kappa_m}}^{l_{\mathbf{i}_1},\cdots,l_{\mathbf{i}_{\kappa_m}}} \hat{\otimes} V_{\mathbf{j}_1,\cdots,\mathbf{j}_{\tilde{\kappa}_m}}^{\mu_{\mathbf{j}_1},\cdots,\mu_{\mathbf{j}_{\tilde{\kappa}_m}}} (p_1^{\otimes q_1}, \cdots, p_m^{\otimes q_m})) . \end{aligned} \tag{3.18}$$

Comparing the coefficient of $u_1^{q_1}\cdots u_m^{q_m}$, we have

$$\prod_{k=1}^m I_{q_k}(p_k^{\otimes q_k}) = \sum_{\substack{\mathbf{j}_1,\cdots,\mathbf{j}_{\tilde\kappa_m}\in S\\ \mathbf{i}_1,\cdots,\mathbf{i}_{\kappa_m}\in\Upsilon}} \sum_{\tilde\chi(k,\vec{l},\vec{\mu})=q_k,k=1,\dots,m} \frac{q_1!\cdots q_m!}{l_{\mathbf{i}_1}!\cdots l_{\mathbf{i}_{\kappa_m}}!\mu_{\mathbf{j}_1}!\cdots\mu_{\mathbf{j}_{\tilde\kappa_m}}!}$$
$$I_n(\hat\otimes_{\mathbf{i}_1,\cdots,\mathbf{i}_{\kappa_m}}^{l_{\mathbf{i}_1},\cdots,l_{\mathbf{i}_{\kappa_m}}}\hat\otimes V_{\mathbf{j}_1,\cdots,\mathbf{j}_{\tilde\kappa_m}}^{\mu_{\mathbf{j}_1},\cdots,\mu_{\mathbf{j}_{\tilde\kappa_m}}}(p_1^{\otimes q_1},\cdots,p_m^{\otimes q_m}))\,. \tag{3.19}$$

Notice that when $|\mathbf{j}|=1$, namely, $\mathbf{j}=(k), k=1,\cdots,m$, then $V_{\mathbf{j}}^{\mu}(f_1,\cdots,f_m)=f_1\hat\otimes\cdots\hat\otimes f_m$. We can separate these terms from the remaining ones, which will satisfy $|\mathbf{j}|\ge 2$. Thus, the remaining multi-indices $\mathbf{j}$'s consists of the set Υ. We can write a multi-index $\vec{\mu}: S\to\mathbb{Z}_+$ as $\vec{\mu}=(n_{(1)},\cdots,n_{(m)},\vec{n})$, where $\vec{n}\in\Upsilon$. We also observe $q_k=\tilde\chi(k,\vec{l},\vec{\mu})=n_{(k)}+\chi(k,\vec{l},\vec{n})$. After replacing $\vec{\mu}$ by $\vec{n}$, (3.19) gives (2.12). This proves Theorem 2.2 for $f_k=p_k^{\otimes q_k}$, $k=1,\cdots,m$. By polarization technique (see e.g. [2, Section 5.2]), we also know the identity (2.12) holds true for $f_k=p_{k,1}\otimes\cdots\otimes p_{k,q_k}$, $p_{k,q_k}\in L^2([0,T]\times\mathbb{R}_0, ds\times\nu(dz))$, $k=1,\cdots,m$. Because both sides of (2.12) are multi-linear with respect to f_k, we know (2.12) holds true for

$$f_k=\sum_{\ell=1}^{\nu_k} c_{k,\ell}p_{k,1,\ell}\otimes\cdots\otimes p_{k,q_k,\ell}\,,\qquad k=1,\cdots,m\,,$$

where $c_{k,\ell}$ are constants, $p_{k,k',\ell}\in L^2([0,T]\times\mathbb{R}_0, ds\times\nu(dz))$, $k=1,\cdots,m, k'=1,\cdots,q_k$ and $\ell=1,\cdots,\nu_k$. Finally, the identity (2.12) is proved by a routine limiting argument.

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References

1. Applebaum, D.: *Lévy Processes and Stochastic Calculus*, Second edition. Cambridge Studies in Advanced Mathematics, 116. Cambridge University Press, Cambridge, 2009.
2. Hu, Y.: *Analysis on Gaussian Spaces*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017.
3. Lee, Y.-J. and ; Shih, H.-H.: The product formula of multiple Lévy–Itô integrals. *Bull. Inst. Math. Acad. Sinica* **32** (2004), no. 2, 71–95.
4. Major, P.: *Multiple Wiener–Itô Integrals. With applications to limit theorems*, Second edition. Lecture Notes in Mathematics, 849. Springer, Cham, 2014.
5. Meyer, P. A.: *Quantum Probability for Probabilists*, Lecture Notes in Mathematics, 1538. Springer-Verlag, Berlin, 1993.
6. Nualart, D.: *The Malliavin Calculus and Related Topics*, Second edition. Probability and its Applications (New York). Springer-Verlag, Berlin, 2006.
7. Protter, P. E.: *Stochastic Integration and Differential Equations*, Second edition. Version 2.1. Corrected third printing. Stochastic Modelling and Applied Probability, 21. Springer-Verlag, Berlin, 2005.
8. Sato, K.: *Lévy Processes and Infinitely Divisible Distributions*, Translated from the 1990 Japanese original. Revised edition of the 1999 English translation. Cambridge Studies in Advanced Mathematics, 68. Cambridge University Press, Cambridge, 2013.
9. Shigekawa, I.: Derivatives of Wiener functionals and absolute continuity of induced measures. *J. Math. Kyoto Univ* **20** (1980), no. 2, 263–289.

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