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CONTINUOUS DEPENDENCE ON THE COEFFICIENTS FOR MEAN-FIELD FRACTIONAL STOCHASTIC DELAY EVOLUTION EQUATIONS

BRAHIM BOUFOUSSI AND SALAH HAJJI*

Abstract. We prove that the mild solution of a mean-field stochastic functional differential equation, driven by a fractional Brownian motion in a Hilbert space, is continuous in a suitable topology, with respect to the initial datum and all coefficients.

1. Introduction

In this paper we are concerned by the following stochastic delay differential equation of McKean-Vlasov type:

$$\begin{array}{l} dx(t) = (Ax(t) + f(t, x_t, P_{x(t)}))dt + g(t)dB^H(t), \quad t \in [0, T] \\ x(t) = \varphi(t), t \in [-r, 0], \end{array} \tag{1.1}$$

where A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators, $(S(t))_{t\geq 0}$, in a Hilbert space X, the driving process B^H is a fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$, $P_{x(t)}$ is the probability distribution of $x(t)$, $x_t \in \mathcal{C}([-r, 0], X)$ is the function defined by $x_t(s) = x(t + s)$ for all $s \in [-r, 0]$, and $f : [0, +\infty) \times \mathcal{C}([-r, 0], X) \times \mathcal{P}_2(X) \to X$, $g : [0, +\infty) \to \mathcal{L}_2^0(Y, X)$ are appropriate functions. Here $\mathcal{P}_2(X)$ denotes the space of probability measures with finite second order moment. That is for each $\mu \in \mathcal{P}_2(X)$, $\int |x|^2 \mu(dx) < \infty$ and $\mathcal{L}_2^0(Y, X)$ denotes the space of all Q-Hilbert-Schmidt operators from Y into X (see section 2 below).

The purpose of this work is twofold, on one hand to prove existence and uniqueness of the mild solution to Eq.(1.1), on the other to provide, in an appropriate topology, sufficient conditions for the continuity of the map $(\varphi, A, f, g) \to x$, where x denotes the mild solution to Eq(1.1).

The most important feature of Equation (1.1) is the dependence of the nonlinearity on the state distribution. Stochastic equations of type (1.1), in finite or infinite dimension, have recieved, during the past decades, a great deal of attention. Because they appear as a mathematical modeling to various dynamical systems involving interactions of a great number of particles (refering to quantum mechanic or statistical physics). Roughly speaking, Eq (1.1) can be seen as a mathematical

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model for large population particle systems which possess the path-dependence in microscopic level:

$$\begin{aligned} &dX^{k,M}(t) = (AX^{k,M}(t) + f(t, X_t^{k,M}, \mu_M(t))dt + g(t)dB^H(t), \quad t \in [0,T] \\ &X^{k,M}(t) = \varphi(t), t \in [-r,0], \end{aligned} \tag{1.2}$$

The coefficient of particle $k \in \{1, 2, \ldots, M\}$ depends on the state of particle k and also on the current empirical distribution of all particle locations, that is $\mu_M(t) = \frac{1}{M}\sum_{1}^{M} \delta_{X^{k,M}(t)}$. In the finite dimensional situations ($X = R^d$), delayed stochastic differential equations (1.1) driven by Wiener process, are obtained naturally as a weak limit of $X^{k,M}$ by letting the particle number $M \to \infty$. We can refer to some relevant works: for example to Budhiraja *et al* [6] where the large deviation of stochastic delays mean-field is considered. Delayed neural network with fractional noise is investigated in Touboul [20]. We also refer to Lasry and Lions[14] for applications in game theory.

The continuous dependence problem we consider here, is motivated by several considerations, first, it provide a justification for the use of numerical schemes, where necessarily one replaces continuous objects by discretized approximations. Furthermore, approximating the unbounded operator A by a bounded one (such as e.g. the Yosida approximation) is a powerful technique to obtain estimates for mild solutions to SPDEs.

We would like to mention that this problem for the stochastic differential equations without delay have been studied intensively. Da Prato and Zabczyk [9] studied the dependence of the solution on the initial datum ξ, Marinelli *et al* [15, 16] studied the problem for the case of Poisson noise. The dependence of the solution on the coefficients f and g were considered by Peszat and Zabczyk [18] and Seidler [19]. The dependence of the solution on A, f, g and φ were considered by Brzeźniak [5], Kunze and Neerven[12, 13]. Govindan and Ahmed [10] studied the Yosida approximations of mild solutions of a semilinear McKean-Vlasov type stochastic evolution equation in a real Hilbert space.

However, in the case of delay, As far as we know, there exist only a few papers published in this field. In Ahmed *et al* [1], the authors discuss the dependence on the initial conditions of the solution of a stochastic functional differential equations with discontinuous data, driven by a Brownian motion, in a finite dimensional space. In [4], the authors studied the dependence on the initial condition for the mild solution of a functional differential equation in Hilbert space driven by a fBm.

To the best of our knowledge, the dependence on the coefficients A, f, g and φ of the solution of a mean-field SFDEs, driven by a fBm in a Hilbert space, has not been considered before in the literature and the aim of this paper is to close this gap.

The rest of this paper is organized as follows, In Section 2 we introduce some notations, concepts, and basic results about fractional Brownian motion, Wiener integral over Hilbert spaces and we recall some preliminary results about Wasserstein distance. The existence and uniqueness of mild solutions are discussed in Section 3 by using Banach fixed point theorem. In Section 4, we investigate the dependence on the coefficients A, f, g and the initial data φ for the mild solution of Equation (1.1).

2. Preliminaries

In this section we collect some notions, conceptions and lemmas on Wiener integrals with respect to an infinite dimensional fractional Brownian motion and we recall some basic results about Wasserstein distance which will be used throughout the whole of this paper.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Consider a time interval $[0, T]$ with arbitrary fixed horizon T and let $\{\beta^H(t), t \in [0, T]\}$ the one-dimensional fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. This means by definition that β^H is a centered Gaussian process with covariance function:

$$R_H(s,t) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

Moreover β^H has the following Wiener integral representation:

$$\beta^H(t) = \int_0^t K_H(t,s) d\beta(s) \tag{2.1}$$

where $\beta = \{\beta(t),\ t \in [0, T]\}$ is a Wiener process, and $K_H(t; s)$ is the kernel given by

$$K_H(t,s) = c_H\, s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du$$

for $t > s$, where $c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}}$ and $\beta(,)$ denotes the Beta function. We put $K_H(t, s) = 0$ if $t \leq s$.

We will denote by $\mathcal{H}$ the reproducing kernel Hilbert space of the fBm. In fact $\mathcal{H}$ is the closure of set of indicator functions $\{1_{[0;t]}, t \in [0, T]\}$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R_H(t,s).$$

The mapping $1_{[0,t]} \to \beta^H(t)$ can be extended to an isometry between $\mathcal{H}$ and the first Wiener chaos and we will denote by $\int_0^T \varphi(s) d\beta^H(s)$ the image of φ by the previous isometry.

Let us consider the operator K_H^* from $\mathcal{H}$ to $L^2([0, T])$ defined by

$$(K_H^* \varphi)(s) = \int_s^T \varphi(r) \frac{\partial K}{\partial r}(r,s) dr$$

We refer to Nualart.[17] for the proof of the fact that K_H^* is an isometry between $\mathcal{H}$ and $L^2([0, T])$. Moreover for any $\varphi \in \mathcal{H}$, we have

$$\int_0^T \varphi(t) d\beta^H(t) = \int_0^T (K_H^* \varphi)(t) d\beta(t) \tag{2.2}$$

Let X and Y be two real, separable Hilbert spaces and let $\mathcal{L}(Y, X)$ be the space of bounded linear operator from Y to X. For the sake of convenience, we shall use the same notation to denote the norms in X, Y and $\mathcal{L}(Y, X)$. Let $Q \in \mathcal{L}(Y, Y)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $trQ = \sum_{n=1}^{\infty} \lambda_n < \infty$. where $\lambda_n \geq 0$ $(n = 1, 2...)$ are non-negative real numbers and $\{e_n\}$ $(n = 1, 2...)$ is a complete orthonormal basis in Y. We define the infinite dimensional fBm on Y with covariance Q as

$$B^H(t) = B^H_Q(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n \beta^H_n(t)$$

where β^H_n are real, independent fBm's.

In order to define Wiener integrals with respect to the Q-fBm, we introduce the space $\mathcal{L}_2^0 := \mathcal{L}_2^0(Y,X)$ of all Q-Hilbert-Schmidt operators $\psi : Y \to X$. We recall that $\psi \in \mathcal{L}(Y,X)$ is called a Q-Hilbert-Schmidt operator, if

$$\|\psi\|^2_{\mathcal{L}_2^0} := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\psi e_n\|^2 < \infty$$

and that the space $\mathcal{L}_2^0$ equipped with the inner product $\langle \varphi, \psi \rangle_{\mathcal{L}_2^0} = \sum_{n=1}^{\infty} \langle \varphi e_n, \psi e_n \rangle$ is a separable Hilbert space.

Now, let $\phi(s)$; $s \in [0,T]$ be a function with values in $\mathcal{L}_2^0(Y,X)$, The Wiener integral of ϕ with respect to B^H is defined by

$$\int_0^t \phi(s)dB^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n}\phi(s)e_n d\beta^H_n(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n}(K^*_H(\phi e_n)(s)d\beta_n(s) \tag{2.3}$$

where β_n is the standard Brownian motion used to present β^H_n as in (2.1) and we have the following result (see Boufoussi and Hajji. [3])

Lemma 2.1. *If $\psi : [0,T] \to \mathcal{L}_2^0(Y,X)$ satisfies $\int_0^T \|\psi(s)\|^2_{\mathcal{L}_2^0} ds < \infty$ then the above sum in* (2.3) *is well defined as a X-valued random variable and we have*

$$\mathbb{E}\| \int_0^t \psi(s)dB^H(s)\|^2 \leq 2Ht^{2H-1} \int_0^t \|\psi(s)\|^2_{\mathcal{L}_2^0} ds$$

We now introduce a class of metrics which we will use in this paper. For any $p \geq 1$, we denote by $\mathcal{P}_p(X)$ the subspace of $\mathcal{P}(X)$ of the probability measures of order p.

For any $p \geq 1$ and $\mu, \nu \in \mathcal{P}_p(X)$, the p-Wasserstein distance $W_p(\mu,\nu)$ is defined by:

$$\begin{aligned} W_p(\mu,\nu) &= \inf \Bigg\{ \left[\int_{X\times X} |x-y|^p \pi(dx,dy) \right]^{1/p}; \\ &\qquad \pi \in \mathcal{P}_p(X\times X) \text{ with marginals } \mu \text{ and } \nu \Bigg\} \end{aligned}$$

Notice that if x and x' are random variables of order p, then

$$W_p\left(P_x, P_{x'}\right) \leq \left(E|x-x'|^p\right)^{1/p}.$$

Notice also that Hölder's inequality implies that:

$$W_p(\mu,\nu) \leq W_q(\mu,\nu),\ \mu,\nu \in \mathcal{P}_p(X), 1 \leq p \leq q$$

The following characterization of the 1-Wasserstein distance W_1 is a direct consequence of the Kantorovich duality theorem. See [Carmona and Delarue. [8], Corollary 5. 4].

Lemma 2.2. *If* (E,d) *is a complete separable metric space, and* $\mu,\nu \in \mathcal{P}_1(E)$, *then,*

$$W_1(\mu,\nu) = \sup_{\phi:|\phi(x)-\phi(y)|\leq d(x,y)} \int_E \phi(x)(\mu-\nu)(dx).$$

where the supremum is taken over all the real valued continuous functions

3. Existence and Uniqueness Result

In this section, we study the existence and uniqueness of mild solutions for Equation (1.1). For this equation we make the following assumption on the coefficients:

($\mathcal{A}$) The operator A generates a strongly continuous analytic semigroup $(S(t))_{t\geq 0}$, on X.

We recall that a closed operator A generates a strongly continuous analytic semigroup on X if and only if A is densely defined and sectorial, i.e., there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\{\lambda \in \mathbb{C} : \Re e\, \lambda > \omega\}$ is contained in the resolvent set $\rho(A)$ and $\sup_{\Re e\, \lambda > \omega} \| (\lambda-\omega)R(\lambda, A) \| \leq M$. In this context, we say that A is sectorial of type (M,ω).

Concerning f and g, we shall assume:

($\mathcal{F}$) The function $f : [0,T] \times \mathcal{C}([-r,0],X) \times \mathcal{P}_2(X) \to X$ satisfies the following conditions:

(i) There exists a positive constant $C_f > 0$ such that, for all $t \in [0,T]$ $x,y \in \mathcal{C}([-r,T],X)$ and μ, $\nu \in \mathcal{C}([0,T],\mathcal{P}_2(X))$

$$\int_0^t \|f(s,x_s,\mu(s)) - f(s,y_s,\nu(s))\|^2 ds \leq C_f \left(\int_{-r}^t \|x(s)-y(s)\|^2 ds + \int_0^t W_2^2(\mu(s),\nu(s)) ds \right)$$

(ii) For arbitrary $\mu \in \mathcal{C}([0,T],\mathcal{P}_2(X))$, there exists a positive constant $\widetilde{C}_f = \widetilde{C}_f(\mu) > 0$ such that

$$\int_0^T \|f(s,0,\mu(s))\|^2 ds \leq \widetilde{C}_f.$$

($\mathcal{G}$) The function $g : [0,+\infty) \to \mathcal{L}_2^0(Y,X)$ satisfies

$$\int_0^T \|g(s)\|^2_{\mathcal{L}_2^0} ds < \infty.$$

Moreover, we assume that $\varphi \in \mathcal{C}([-r,0],\mathbb{L}^2(\Omega,X))$.

Definition 3.1. A X-valued process $\{x(t),\ t \in [-r,T]\}$, is called a mild solution of equation (1.1) if

$i)$ $x(.) \in \mathcal{C}([-r,T],\mathbb{L}^2(\Omega,X))$,

$ii)$ $x(t) = \varphi(t)$, $-r \leq t \leq 0$.

iii) For arbitrary $t \in [0, T]$, we have

$$\begin{aligned} x(t) &= S(t)\varphi(0) + \int_0^t S(t-s)f(s, x_s, \mu(s))ds \\ &+ \int_0^t S(t-s)g(s)dB(s) \quad \mathbb{P} - a.s \end{aligned}$$

We can now state the main result of this section.

Theorem 3.2. *Suppose that* $(\mathcal{A}), (\mathcal{F})$ *and* $(\mathcal{G})$ *are satisfied. Then for every* $\varphi \in \mathcal{C}([-r, 0], \mathbb{L}^2(\Omega, X))$*, Eq.*(1.1) *has a unique mild solution on* $[-r, T]$.

Proof. Let $\mu \in \mathcal{C}([0, T], \mathcal{P}_2(X))$ be temporarily fixed. The classical existence result for stochastic functional differential equations (cf. [7]) guarantees the existence and uniqueness of a mild solution of the equation

$$\begin{aligned} &dx(t) = (Ax(t) + f(t, x_t, \mu(t)))dt + g(t)dB^H(t), \quad t \in [0, T] \\ &x(t) = \varphi(t), t \in [-r, 0], \end{aligned}$$

We denote its solution by x_μ.

Let $\mathcal{L}(x_\mu) = \{\mathcal{L}(x_\mu(t)) : t \in [0, T]\}$ denote the law of x_μ and define an operator ψ on $\mathcal{C}([0, T], \mathcal{P}_2(X))$ by $\psi(\mu) = \mathcal{L}(x_\mu)$. Then it is clear that to prove the existence of mild solutions to equation (1.1) is equivalent to find a fixed point for the operator ψ. Next we will show by using Banach fixed point theorem that ψ has a unique fixed point. We divide the subsequent proof into two steps.

Step 1.

For arbitrary $\mu \in \mathcal{C}([0, T], \mathcal{P}_2(X))$, let us prove that $t \to \psi(\mu)(t)$ is continuous on the interval $[0, T]$.

Let $0 < t < T$ and $|h|$ be sufficiently small. Then, we have

$$\begin{aligned} W_2^2(\psi(\mu)(t+h), \psi(\mu)(t)) &= W_2^2(\mathbb{P}_{x_\mu(t+h)}, \mathbb{P}_{x_\mu(t)}) \\ &\leq E \parallel x_\mu(t+h) - x_\mu(t) \parallel^2 \to 0 \quad \text{as } h \to 0 \end{aligned}$$

Hence, we conclude that the function $t \to \psi(\mu)(t)$ is continuous on $[0, T]$.

Step 2.

Now, we are going to show that ψ is a contraction mapping in $\mathcal{C}([0, T_1], \mathcal{P}_2(X))$ with some $T_1 \leq T$ to be specified later.

Let $\mu, \nu \in \mathcal{C}([0, T], \mathcal{P}_2(X))$. By Lipschitz property of f combined with Hölder's inequality, we obtain for any fixed $t \in [0, T]$

$$\begin{aligned} \|x_\mu(t) - x_\nu(t)\|^2 &= \| \int_0^t S(t-s)(f(s, (x_\mu)_s, \mu(s)) - f(s, (x_\nu)_s, \nu(s)))ds\|^2 \\ &\leq tM^2 \int_0^t \|f(s, (x_\mu)_s, \mu(s)) - f(s, (x_\nu)_s, \nu(s))\|^2 ds \\ &\leq tM^2 C_f \int_{-r}^t \|x_\mu(s) - x_\nu(s)\|^2 ds \\ &\quad + tM^2 C_f \int_0^t W_2^2(\mu(s), \nu(s))ds \end{aligned}$$

Then, Gronwall's lemma implies that

$$\sup_{s\in[0,t]} \mathbb{E}\|x_\mu(s) - x_\nu(s)\|^2 \leq \gamma(t) \sup_{s\in[0,t]} W_2^2(\mu(s), \nu(s)).$$

where

$$\gamma(t) = t^2 M^2 C_f \exp\left(\frac{t^2}{2} M^2 C_f\right) \text{ and } M = \sup_{0\leq t\leq T} \|S(t)\|$$

Hence, it follows that

$$\sup_{s\in[0,t]} W_2^2(\psi(\mu)(s), \psi(\nu)(s)) \leq \gamma(t) \sup_{s\in[0,t]} W_2^2(\mu(s), \nu(s)).$$

Since $\gamma(0) = 0 < 1$. Then there exists $0 < T_1 \leq T$ such that $0 < \gamma(T_1) < 1$ and ψ is a contraction mapping on $\mathcal{C}([0, T_1], \mathcal{P}_2(X))$ and therefore has a unique fixed point μ, then, x_μ is a mild solution of equation (1.1) on $[-r, T_1]$. This procedure can be repeated in order to extend the solution to the entire interval $[-r, T]$ in finitely many steps. This completes the proof. □

3.1. Example. Let us consider the system with distributed delay

$$\begin{aligned} dx(t) &= (Ax(t) + \int_{-r}^{0} F_1(t, s, x(t+s))ds + \int_X F_2(z) P_{x(t)}(dz) F_3(t))dt \\ &\quad + g(t) dB^H(t), \quad t \in [0, T] \\ x(t) &= \varphi(t), \quad t \in [-r, 0], \end{aligned} \tag{3.1}$$

where $r > 0$. Let us assume the previous hypotheses on operators A, g and the fractional Brownian motion, and assume that $F_1 : [0, +\infty) \times [-r, 0] \times X \to X$ is a measurable function such that

$$\|F_1(t, s, x) - F_1(t, s, y)\| \leq C\|x - y\|, \forall x, y \in X, t \geq 0, s \in [-r, 0]$$

where C is a non-negative constant, and $F_2 : X \to \mathbb{R}$ is a 1-Lipschitz continuous function. $F_3 : [0, +\infty) \to X$ is a measurable and bounded function.

Observe that our new problem (3.1) can be re-written in our abstract functional formulation by defining a new function $f : [0, T] \times \mathcal{C}([-r, 0], X) \times \mathcal{P}_2(X) \to X$ as follows

$$f(t, \xi, \mu) = \int_{-r}^{0} F_1(t, s, \xi(s))ds + \int_X F_2(z)\mu(dz) F_3(t)$$

Let us show that f satisfies conditions (i) and (ii) in the hypothesis $(\mathcal{F})$.

For $t \in [0, T]$, $x, y \in \mathcal{C}([-r, T], X)$ and μ, $\nu \in \mathcal{C}([0, T], \mathcal{P}_2(X))$, we have

$$\begin{aligned} \|f(t, x_t, \mu(t)) &- f(t, y_t, \nu(t))\|^2 \\ &\leq 2\|\int_{-r}^{0} (F_1(t, s, x(t+s)) - F_1(t, s, y(t+s)))ds\|^2 \\ &\quad + 2\|\int_X F_2(z)(\mu(t) - \nu(t))(dz) F_3(t)\|^2 \end{aligned} \tag{3.2}$$

By Lipschitz property of F_1 combined with Cauchy Schwarz inequality, we obtain

$$\begin{aligned}\|\int_{-r}^{0}(F_1(t,s,x(t+s)) &- F_1(t,s,y(t+s)))ds\|^2 \\ &\leq r\int_{-r}^{0}C^2\|x(t+s)-y(t+s)\|^2ds \\ &\leq rC^2\int_{-r}^{t}\|x(s)-y(s)\|^2ds \qquad (3.3)\end{aligned}$$

By Lemma 2.2, we obtain that

$$\begin{aligned}\|\int_X F_2(z)(\mu(t)-\nu(t))(dz)F_3(t)\|^2 &\leq W_1^2(\mu(t),\nu(t)) \parallel F_3 \parallel_\infty^2 \\ &\leq W_2^2(\mu(t),\nu(t)) \parallel F_3 \parallel_\infty^2 \qquad (3.4)\end{aligned}$$

Inequalities (3.2), (3.3) and (3.4) imply that:

$$\begin{aligned}\int_0^t \|f(s,x_s,\mu(s))-f(s,y_s,\nu(s))\|^2ds &\leq 2rC^2T\int_{-r}^{t}\|x(s)-y(s)\|^2ds \\ &+2 \parallel F_3 \parallel_\infty^2 \int_0^t W_2^2(\mu(s),\nu(s))ds\end{aligned}$$

The last inequality implies that the function f satisfies the condition (i) in hypothesis $(\mathcal{F})$.

The second condition (ii) is trivial. Consequently, all the hypotheses in Theorem 3.2 are satisfied and we can ensure the existence and uniqueness of the mild solution of the system (3.1) on the interval $[-r,T]$.

4. Continuous Dependence on the Coefficients

In this section, we present and prove our main result concerning the dependence on the coefficients A, f, g and the initial datum for the mild solution to equation (1.1). Let us consider for each $n \in \mathbb{N}$, the equation

$$\begin{array}{l} dx(t) = (A_nx(t) + f_n(t,x_t,P_{x(t)}))dt + g_n(t)dB^H(t), \quad t \in [0,T] \\ x(t) = \varphi_n(t), t \in [-r,0], \end{array} \qquad (4.1)$$

To establish our main result we need to impose the following assumptions:

- $(\mathcal{A}_1)$ The operators A and A_n are densely defined, closed, and uniformly sectorial on X in the sense, there exist numbers $M \geq 1$ and $\omega \in \mathbb{R}$ such that A and each A_n is sectorial of type (M,ω) .
- $(\mathcal{A}_2)$ The operators A_n converge to A in the strong resolvent sense:

$$\lim_{n\to\infty} R(\lambda,A_n)x = R(\lambda,A)x \;\text{ for all }\; \Re e\,\lambda > \omega \;\text{ and } x \in X$$

Remark 4.1. One of the classical example of A_n is the Yosida approximation. In fact, let us consider an operator A which generates a strongly continuous analytic semigroup $(S(t))_{t\geq 0}$, on X. If we take its Yosida approximands defined by $A_n := n^2R(n,A) - n$. Then assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ hold for these operators. See [Haase.[11], Proposition 2.1.1] and [Arendt *et al* [2], Section 3.6].

Remark 4.2. Under $(\mathcal{A}_1)$, the operators A and A_n generate strongly continuous analytic semigroups S, S_n satisfying the uniform bounds $\|S(t)\|, \|S_n(t)\| \leq Me^{\omega t}, t \geq 0$.

The following Trotter-Kato type approximation theorem is well known; see [[2], Theorem 3.6.1]

Lemma 4.3. *Under condition $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, we have:*
For all $t \geq 0$ and $x \in X$, $S_n(t)x \to S(t)x$ as $n \to \infty$ and the convergence is uniform on compact subsets of $[0,\infty) \times X$.

For the nonlinearities f and f_n, we will make the following assumptions:

- $(\mathcal{F}_1)$ The maps $f, f_n : [0,T] \times \mathcal{C}([-r,0],X) \times \mathcal{P}_2(X) \to X$ satisfy condition $(\mathcal{F})$ with uniform constants C_f and $\widetilde{C}_f$.
- $(\mathcal{F}_2)$ For all $x \in \mathcal{C}([-r,T],X)$ and $\mu \in \mathcal{C}([0,T],\mathcal{P}_2(X))$,

$$\lim_{n\to\infty} \int_0^T |f_n(s,x_s,\mu(s)) - f(s,x_s,\mu(s))|^2 ds = 0$$

For the functions g and g_n, we make the following assumptions:

- $(\mathcal{G}_1)$ The maps $g, g_n : [0,+\infty) \to \mathcal{L}_2^0(Y,X)$ satisfy

$$\int_0^T \|g_n(s)\|^2_{\mathcal{L}_2^0} ds, \int_0^T \|g(s)\|^2_{\mathcal{L}_2^0} ds < \infty.$$

- $(\mathcal{G}_2)$

$$\lim_{n\to\infty} \int_0^T \|g_n(s) - g(s)\|^2_{\mathcal{L}_2^0}\, ds = 0$$

The main result of this paper is given in the next theorem.

Theorem 4.4. *Suppose that the operators A and A_n satisfy $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, the nonlinearities f and f_n satisfy $(\mathcal{F}_1)$ and $(\mathcal{F}_2)$, the functions g and g_n satisfy $(\mathcal{G}_1)$ and $(\mathcal{G}_2)$ and the initial data φ and φ_n satisfy $\varphi_n \to \varphi$ in $\mathcal{C}([-r,0],\mathbb{L}^2(\Omega,X))$ and let x and x^n be the mild solutions to (1.1) and (4.1), respectively. Then, $x^n \to x$ in $\mathcal{C}([-r,T],\mathbb{L}^2(\Omega,X))$ as $n \to \infty$, that is :*

$$\lim_{n\to\infty} \sup_{-r\leq t\leq T} E \mid x^n(t) - x(t) \mid^2 = 0$$

Proof. Let $t \in [0,T]$ and $0 \leq u \leq t$. By the triangle inequality one has

$$\begin{aligned} E \mid x^n(u) &- x(u) \mid^2 \\ &\leq 3E \mid S_n(u)\varphi_n(0) - S(u)\varphi(0) \mid^2 \\ &\quad +3E \mid \int_0^u S_n(u-s)f_n(s,x_s^n,P_{x^n(s)}) - S(u-s)f(s,x_s,P_{x(s)})ds \mid^2 \\ &\quad +3E \mid \int_0^u S_n(u-s)g_n(s) - S(u-s)g(s)dB(s) \mid^2 \\ &\leq 3(I_1 + I_2 + I_3) \end{aligned} \tag{4.2}$$

We estimate the various terms of the right hand side separately.

For the first term, by using the fact that the operator norm of $S_n(t)$ is bounded by $Me^{\omega T}$ we get that

$$\begin{aligned} I_1 &\leq 2E \mid (S_n(u)-S(u))\varphi(0) \mid^2 + 2E \mid S_n(u)(\varphi(0)-\varphi_n(0)) \mid^2 \\ &\leq 2E \sup_{0\leq u\leq T} \mid (S_n(u)-S(u))\varphi(0) \mid^2 + 2M^2e^{2\omega T} \sup_{-r\leq s\leq 0} E \mid \varphi(s)-\varphi_n(s) \mid^2 \end{aligned}$$

Moreover, by Trotter-Kato's theorem, $S_n(.)\varphi(0)$ converges to $S(.)\varphi(0)$ $\mathbb{P}$-a.s. uniformly on compact sets, i.e.

$$\lim_{n\to\infty} \sup_{0\leq u\leq T} \mid (S_n(u)-S(u))\varphi(0) \mid = 0$$

which implies, together with

$$\sup_{0\leq u\leq T} \mid (S_n(u)-S(u))\varphi(0) \mid^2 \leq 4M^2e^{2\omega T} \mid \varphi(0) \mid^2$$

and $E \mid \varphi(0) \mid^2 < \infty$, that $\lim_{n\to\infty} E \sup_{0\leq u\leq T} \mid (S_n(u)-S(u))\varphi(0) \mid^2 = 0$, thanks to the dominated convergence theorem. Then,

$$\begin{aligned} I_1 \leq \gamma_n &:= 2E \sup_{0\leq u\leq T} \mid (S_n(u)-S(u))\varphi(0) \mid^2 \\ &\quad + 2M^2e^{2\omega T} \sup_{-r\leq s\leq 0} E \mid \varphi(s)-\varphi_n(s) \mid^2 \to 0 \end{aligned} \tag{4.3}$$

For the second term I_2, the triangle inequality yields

$$\begin{aligned} I_2 &\leq 3E \mid \int_0^u S_n(u-s)(f_n(s,x_s,P_{x(s)}) - f_n(s,x_s^n,P_{x^n(s)}))ds \mid^2 \\ &\quad + 3E \mid \int_0^u S_n(u-s)(f(s,x_s,P_{x(s)}) - f_n(s,x_s,P_{x(s)}))ds \mid^2 \\ &\quad + 3E \mid \int_0^u (S_n(u-s)-S(u-s))f(s,x_s,P_{x(s)})ds \mid^2 \\ &\leq 3(I_{21}+I_{22}+I_{23}) \end{aligned} \tag{4.4}$$

By using Cauchy Schwarz inequality and condition $(\mathcal{F}_1)$, we obtain that

$$\begin{aligned} I_{21} &\leq TM^2e^{2\omega T}C_f \left\{ \int_{-r}^u E \mid x(s)-x^n(s) \mid^2 ds + \int_0^u W_2^2(P_{x(s)},P_{x^n(s)})ds \right\} \\ &\leq TM^2e^{2\omega T}C_f \left\{ r \sup_{-r\leq s\leq 0} E \mid \varphi(s)-\varphi_n(s) \mid^2 \right. \\ &\qquad \left. + 2\int_0^u \sup_{-r\leq v\leq s} E \mid x(v)-x^n(v) \mid^2 ds \right\} \end{aligned} \tag{4.5}$$

Cauchy Schwarz inequality implies that

$$I_{22} \leq TM^2e^{2\omega T} \int_0^T E \mid f(s,x_s,P_{x(s)}) - f_n(s,x_s,P_{x(s)}) \mid^2 ds \tag{4.6}$$

The condition $(\mathcal{F}_2)$ assures that

$$\lim_{n\to\infty} \int_0^T \mid f(s,x_s,P_{x(s)}) - f_n(s,x_s,P_{x(s)}) \mid^2 ds = 0$$

The condition $(\mathcal{F}_1)$ assures that

$$\begin{aligned}
&\int_0^T \mid f(s,x_s,P_{x(s)}) - f_n(s,x_s,P_{x(s)}) \mid^2 ds \\
&\leq 4\int_0^T \mid f(s,x_s,P_{x(s)}) - f(s,0,P_{x(s)}) \mid^2 ds + 4\int_0^T \mid f(s,0,P_{x(s)}) \mid^2 ds \\
&+4\int_0^T \mid f_n(s,x_s,P_{x(s)}) - f_n(s,0,P_{x(s)}) \mid^2 ds + 4\int_0^T \mid f_n(s,0,P_{x(s)}) \mid^2 ds \\
&\leq 8C_f \int_{-r}^T \mid x(s) \mid^2 \; ds + 8\widetilde{C}_f \in \mathbb{L}^1(\Omega,\mathbb{P})
\end{aligned}$$

Then we conclude by the Lebesgue dominated theorem that

$$\lim_{n\to\infty}\int_0^T E \mid f(s,x_s,P_{x(s)}) - f_n(s,x_s,P_{x(s)}) \mid^2 ds = 0$$

In order to estimate the term I_{23}, let us denotes $S*h(t) = \int_0^t S(t-s)h(s)ds$, then,

$$\int_0^u (S_n(u-s) - S(u-s))f(s,x_s,P_{x(s)})ds = S_n * \widetilde{f}(u) - S * \widetilde{f}(u)$$

where $\widetilde{f}(u) = f(u,x_u,P_{x(u)})$.

By Theorem 3.1 in [16], we have

$$\lim_{n\to\infty}\sup_{u\leq T} \mid S_n * \widetilde{f}(u) - S * \widetilde{f}(u) \mid = 0$$

and since

$$\sup_{u\leq T} \mid S_n * \widetilde{f}(u) - S * \widetilde{f}(u) \mid^2 \leq 4TM^2e^{\omega T}\int_0^T |f(s,x_s,P_{x(s)})|^2 \, ds \in \mathbb{L}^1(\Omega,\mathbb{P})$$

Then we conclude by the Lebesgue dominated theorem that

$$\lim_{n\to\infty} E\sup_{u\leq T} \mid S_n * \widetilde{f}(u) - S * \widetilde{f}(u) \mid^2 = 0$$

which implies

$$\lim_{n\to\infty}\sup_{u\leq T} E \mid \int_0^u (S_n(u-s) - S(u-s))f(s,x_s,P_{x(s)})ds \mid^2 = 0 \tag{4.7}$$

Inequalities (4.4), (4.5), (4.6) and (4.7) together imply that:

$$I_2 \leq \delta_n + 6TM^2e^{2T\omega}C_f\int_0^u \sup_{-r\leq v\leq s} E \mid x(v) - x^n(v) \mid^2 ds \tag{4.8}$$

where

$$\begin{aligned}
\delta_n \quad := \quad & 3rTM^2e^{2\omega T}C_f \sup_{-r\leq s\leq 0} E \mid \varphi(s) - \varphi_n(s) \mid^2 \\
& +3\sup_{u\leq T} E \mid \int_0^u (S_n(u-s) - S(u-s))f(s,x_s,P_{x(s)})ds \mid^2 \\
& +3TM^2e^{2T\omega}\int_0^T E \mid f(s,x_s,P_{x(s)}) - f_n(s,x_s,P_{x(s)}) \mid^2 ds
\end{aligned}$$

with $\lim_{n\to\infty} \delta_n = 0$

For the last term I_3, by Lemma 2.1, we have

$$\begin{aligned}
I_3 &\leq 2HT^{2H-1}\int_0^u \mid S(u-s)g(s) - S_n(u-s)g_n(s) \mid^2 ds \\
&\leq 4HT^{2H-1}\left\{\int_0^u \mid S_n(u-s)(g(s)-g_n(s)) \mid^2 ds \right. \\
&\qquad \left. + \int_0^u \mid (S(u-s)-S_n(u-s))g(s) \mid^2 ds\right\} \\
&\leq 4HT^{2H-1}(I_{31}+I_{32}) \qquad (4.9)
\end{aligned}$$

The condition $(\mathcal{G}_2)$ assures that

$$I_{31} \leq M^2 e^{2T\omega}\int_0^T \mid g(s)-g_n(s) \mid^2 ds \to 0 \,\text{as}\, n\to\infty \qquad (4.10)$$

Let us now prove that $\lim_{n\to\infty}\sup_{u\leq T}\int_0^u \mid (S(u-s)-S_n(u-s))g(s) \mid^2 ds = 0.$

In the simplest case $g(s)=\xi$ for some constant $\xi\in X$, we have by Lemma 4.3 that $(S(u-s)-S_n(u-s))\xi \to 0$ as $n\to\infty$ and the convergence is uniform on $[0,T]$, then, it follow that

$$\lim_{n\to\infty}\sup_{u\leq T}\int_0^u \mid (S(u-s)-S_n(u-s))g(s) \mid^2 ds = 0.$$

The same result remain true when g is a simple function.

The general case can be easily proved by approximating g by simple functions then,

$$\lim_{n\to\infty}\sup_{u\leq T}\int_0^u \mid (S(u-s)-S_n(u-s))g(s) \mid^2 ds = 0 \qquad (4.11)$$

Inequalities (4.9), (4.10) and (4.11) together imply that:

$$\begin{aligned}
I_3 \leq \lambda_n &:= 4HT^{2H-1}\left\{\sup_{u\leq T}\int_0^u \mid (S(u-s)-S_n(u-s))g(s) \mid^2 ds\right. \\
&\qquad \left. + M^2e^{2T\omega}\int_0^T \mid g(s)-g_n(s) \mid^2 ds\right\} \qquad (4.12)
\end{aligned}$$

with $\lim_{n\to\infty} \lambda_n = 0$

Thus, inequalities (4.2),(4.3), (4.8) and (4.12) together imply

$$\begin{aligned}
\sup_{0\leq u\leq t} E \mid x^n(u)-x(u) \mid^2 &\leq 3\left(\gamma_n+\delta_n+\lambda_n\right) \\
&\quad + 18TM^2e^{2T\omega}C_f\int_0^t \sup_{-r\leq u\leq s} E \mid x^n(u)-x(u) \mid^2 ds
\end{aligned}$$

Using the fact that $x = \varphi$ and $x^n = \varphi_n$ on $[-r, 0]$ we get

$$\sup_{-r\le u\le t} E \mid x^n(u) - x(u) \mid^2 \le 3(\gamma_n + \delta_n + \lambda_n) + \sup_{-r\le u\le 0} E \mid \varphi_n(u) - \varphi(u) \mid^2 + 18TM^2e^{2T\omega}C_f \int_0^t \sup_{-r\le u\le s} E \mid x^n(u) - x(u) \mid^2 ds$$

which implies, by Gronwall's inequality,

$$\lim_{n\to\infty} \sup_{-r\le u\le T} E \mid x^n(u) - x(u) \mid^2 = 0$$

This completes the proof. □

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