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SCS 18: Continuous Lattices and Universal Algebra

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SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

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TOPIC Continuous Lattices and Universal Algebra

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I'd like to clarify and extend some of Scott's remarks and show how with "algebra" and "abstract nonsense" some of the known results on continuous lattices can be formulated.

1. General Abstract Nonsense Algebra

A monad (or triple) over a category $\mathcal{K}$ is a triple $\mathbb{T} = (T, \eta, \mu)$ where $T : \mathcal{K} \to \mathcal{K}$ is an endofunctor and $\eta : 1_{\mathcal{K}} \to T$, $\mu : T^2 \to T$ are natural transformations satisfying $\forall X \in \mathcal{K}$

$$\begin{tikzcd} TX \arrow[r, "\eta TX"] \arrow[dr, equal] & T^2X \arrow[d, "\mu X"] & TX \arrow[l, "T\eta X"'] \arrow[dl, equal] \\ & TX & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T^3X \arrow[r, "T\mu X"] \arrow[d, "\mu TX"'] & T^2X \arrow[d, "\mu X"] \\ T^2X \arrow[r, "\mu X"] & TX \end{tikzcd}$$

A monad $\mathbb{T}$ over $\mathcal{K}$ naturally determines a category $\mathcal{K}^{\mathbb{T}}$ of $\mathbb{T}$-algebras and their morphisms viz:

West Germany: TH Darmstadt (Gierz, Keimel, Day, Visit.)
U. Tübingen (Mislove, Visit.)

England: U. Oxford (Scott)

USA: U. California, Riverside (Stralka)
LSU Baton Rouge (Lawson)
Tulane U., New Orleans (Hofmann, Mislove)
U. Tennessee, Knoxville (Carruth, Crawley)

$\mathcal{K}^{\Pi}$ Objects (A, α) where $A \in \mathcal{K}$, and $\alpha : TA \to A$

satisfies
$$\begin{tikzcd} A \arrow[r, "\eta A"] \arrow[dr, equal] & TA \arrow[d, "\alpha"] \\ & A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T^2A \arrow[r, "T\alpha"] \arrow[d, "\mu A"] & TA \arrow[d, "\alpha"] \\ TA \arrow[r, "\alpha"] & A \end{tikzcd}$$

$\mathcal{K}^{\Pi}$-maps: $f : (A, \alpha) \to (B, \beta)$, $f \in \mathcal{K}(A, B)$ satisfying
$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "\alpha"] & TB \arrow[d, "\beta"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

$\mathcal{K}^{\Pi}$ is in essence a "variety of algebras" over $\mathcal{K}$ in that

(1) The obvious forgetful functor $U^{\Pi} : \mathcal{K}^{\Pi} \to \mathcal{K}$ preserves + reflects limits and has a left adjoint a "free Π-algebra" functor ~~$\mathcal{F}$~~ $F^{\Pi} : \mathcal{K} \to \mathcal{K}^{\Pi}$ viz:

$F^{\Pi}(X) = (T(X), \mu X)$. $F^{\Pi}(X \xrightarrow{f} Y) = Tf : (TX, \mu X) \to (TY, \mu Y)$

If $f : X \to U^{\Pi}(A, \alpha)$ then $\alpha \circ Tf$ is the unique extension $(TX, \mu X) \to (A, \alpha)$

(2) When $\mathcal{K}$ is sufficiently nice ie $\mathcal{K} = \mathcal{S}$ (sets) one obtains all the homomorphism theorems, product, coproduct constructions etc in the usual algebraic way. Moreover, as described by Manes in [LNM 80], operations and/or polynomials have a description of being natural transformations.

That is: for a set I, an I-ary operation is a natural transformation from $(\ )^I$ into T

The Yoneda Lemma provides a natural equivalence between $n.t(\,(\ )^{I}, T\,)$ and $T(I)$ and therefore the free $\mathbb{T}$-algebra <u>IS</u> the algebra of all I-ary ~~of~~ polynomials. Moreover as Manes showed, for $\mathbb{T}$-algebras (A,α), (B,β) ~~[illegible]~~ and $f : A \to B$ T.F.A.E

(1) $f : (A,\alpha) \longrightarrow (B,\beta)$

(2) $\forall I$ $\forall$ I-ary poly $p : (\ )^{I} \to T$

$$\begin{tikzcd} (A)^{I} \arrow[r, "f^{I}"] \arrow[d, "\tilde{p}A"'] & (B)^{I} \arrow[d, "\tilde{p}B"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

(3) $\forall$ A-ary poly $p : (\ )^{A} \to T$, the above diagram holds

where $\tilde{p}A = \left(A^{I} \xrightarrow{pA} T(A) \xrightarrow{\alpha} A \right)$

The monads of interest here are the ultrafilter, filter, and principal filter monads. In all cases the "embedding of the generators" $\eta : 1_S \to T$ attaches the principal (ultra) filter of all subsets containing x $(x \in X)$, and the algebra structure $\mu : T^2 \to T$ on the free algebras gives for each set X: $M \in \mu X(\mathcal{F})$ iff $\{f \in TX : M \in f\} \in \mathcal{F}$

Real life descriptions of these $\mathbb{T}$-algebras are:

	$\mathbb{T}$	$\mathbb{T}$-algebras	$\mathbb{T}$-morphisms
(Manes)	ultrafilter Ω	Compact T_2 Spaces	continuous functions.
(Day)	filter $\mathbb{F}$	Continuous lattices	$\wedge$ and $\vee\!\!\uparrow$ preserving functions
(?)	principal filter $\mathbb{F}_p$	$\wedge$-semi lattices	$\wedge$ preserving functions

§2: Continuous Lattices as Compact T_2 - Semilattices

Because the "generators" and "algebra operation" of Ω is just the restriction from $\mathbb{F}$. (More precisely there is a (pointwise) monic natural transformation $j : \Omega \hookrightarrow F$ that is a morphism of triples), every continuous lattice is a compact T_2-space by examining the following diagram:

$$
\begin{tikzcd}
\Omega^2 X \arrow[rr, "\Omega\xi"] \arrow[dr, "\alpha"] \arrow[dd, "\mu_\Omega X"'] & & \Omega X \arrow[dr, "jX"] \arrow[dd, "\xi\circ jX" near start] & \\
& F^2 X \arrow[rr, "F\xi" near start] \arrow[dd, "\mu_F X" near start] & & FX \arrow[dd, "\xi"] \\
\Omega(X) \arrow[rr, "\xi\circ jX" near end] \arrow[dr] & & X \arrow[dr, equal] & \\
& FX \arrow[rr, "\xi"'] & & X
\end{tikzcd}
$$

where $\alpha = \left(
\begin{tikzcd}
& \Omega F X \arrow[dr, "jFX"] & \\
\Omega^2 X \arrow[ur, "\Omega jX"] \arrow[dr, "j\Omega X"'] & & F^2 X \\
& F\Omega X \arrow[ur, "FjX"'] &
\end{tikzcd}
\right)$

In order to make (X, ξ) a compact T_2 $\wedge$-semilattice, we must also show that $\wedge$ is continuous ie $\cancel{\wedge:(X,\xi)}$ $\wedge : (X, \xi\circ jX)^2 \longrightarrow (X, \xi\circ jX)$ is a Ω-homomorphism.

To do this we must first determine what $\wedge$ is as a 2-ary polynomial for $\mathbb{F}$. It is $\{2\} \in F(2)$ by Yoneda

ie
$$\begin{array}{ccccc} X^2 & \xrightarrow{pX} & F(X) & \xrightarrow{\xi} & X \quad \text{is} \\ (x_1, x_2) & \longmapsto & f_{\{x_1, x_2\}} & \longmapsto & x_1 \wedge x_2 \end{array}$$

where f_M is the principal filter generated by $M \subseteq X$

Continuity in Ω will follow ~~if we can~~ by restriction if we can show this for continuous lattices ~~(illeg.)~~ so consider $(X, \xi) \in \mathcal{S}^{\mathbb{F}}$ and

$$\begin{tikzcd} F(X^2) \arrow[r, "F(\wedge)"] \arrow[d, "\xi^{[2]}"'] & F(X) \arrow[d, "\xi"] \\ X^2 \arrow[r, "\wedge"] & X \end{tikzcd}$$

(Note: Product algebras $(A, \alpha) \times (B, \beta)$ are formed by $T(A \times B) \xrightarrow{\varphi} T(A) \times T(B) \xrightarrow{\alpha \times \beta} A \times B$

where φ is the unique map given by
$$\begin{tikzcd} T(A \times B) \arrow[r, "T(\pi_A)"] \arrow[d, "T(\pi_B)"'] & TA \\ T(B) & \end{tikzcd}$$
)

We compute

$$
\begin{tikzcd}
\mathfrak{f} \arrow[r, mapsto, "F(\wedge)"] \arrow[d, mapsto, "\varphi"'] & \left\{ \{ s_1 \wedge s_2 : (s_1, s_2) \in S \} : S \in \mathfrak{f} \right\}^{\text{filter generated}} \arrow[d, mapsto, "\xi"] \\
\left(\{ \pi_1(S) : S \in \mathfrak{f} \}^{\text{filter gen}}, \{ \pi_2(S) : S \in \mathfrak{f} \}^{\text{filter gen}} \right) \arrow[d, mapsto, "\xi \times \xi"'] & \bigvee_{S \in \mathfrak{f}} \bigwedge_{(s_1, s_2) \in S} (s_1 \wedge s_2) \\
\left(\bigvee_{S \in \mathfrak{f}} \bigwedge_{(s_1, s_2) \in S} s_1 , \bigvee_{T \in \mathfrak{f}} \bigwedge_{(t_1, t_2) \in T} t_2 \right) \arrow[r, mapsto, "\wedge"] & \bigvee_{(S, T) \in \mathfrak{f} \times \mathfrak{f}} \left(\bigwedge_{(s_1, s_2) \in S} s_1 \wedge \bigwedge_{(t_1, t_2) \in T} t_2 \right)
\end{tikzcd}
$$

the last two are equal as filters are down-directed

§3 Comments on Scott SCS (23 Aug 76)

(1) There is an algebraic way to map $\mathrm{Set}(A,B)$ into $\mathrm{Cont}((A,\alpha),(B,\beta))$ that hides the use of $\ll$ however it is not the same as $\gamma(f)(x) = \bigvee\{f(y) : y \ll x\}$ and does <u>not</u> seem to be continuous, namely:

$$f : U^{\mathbb{F}}(A,\alpha) \longrightarrow U^{\mathbb{F}}(B,\beta) \qquad \text{set map}$$

$$\beta \circ Ff \ : \underset{(F(A),\mu A)}{F^{\mathbb{F}} U^{\mathbb{F}}(A,\alpha)} \longrightarrow (B,\beta) \qquad \mathbb{F}\text{-morphism}$$

$$\beta \circ Ff \circ k_{(A,\alpha)} : (A,\alpha) \longrightarrow (B,\beta) \qquad \text{continuous}$$

where $k_{(A,\alpha)} : A \hookrightarrow F(A)$ is the duality map attached to $\alpha : F(A) \longrightarrow A$ which <u>is</u> a homomorphism

computing this gives the function (continuous)

$$\bar{\gamma}(f) : \quad a \longmapsto \bigvee_{d \ll a} \bigwedge \{f(x) : d \le x \le 1\}$$

Now $\gamma(f) = \bar{\gamma}(f)$ for <u>monotone</u> functions but in general $f \mapsto \bar{\gamma}(f)$ seems not to be continuous from $S(A,B)$ to $\mathrm{Cont}((A,\alpha),(B,\beta))$

(2) $\mathcal{S}^{\mathbb{F}}((A,\alpha),(B,\beta))$ will be a retract of $\mathcal{S}(A,B)$ if (A,α) is a projective $\mathcal{S}^{\mathbb{F}}$-algebra. These are the $\mathcal{S}^{\mathbb{F}}$ retracts of the filter lattices, must be distributive, algebraic lattices, and in the finite case correspond to all finite distributive lattices. However don't know a general description.

(3) As the polynomials form a proper class (ie every filter on every ordinal) restricting filters to have a base of sets ~~all~~ all of whose cardinality is $\leq \alpha$ some $\alpha \in \mathrm{Ord}$ will give a different class (with the same algebras up to that cardinal $|\alpha|$) However with the Ω triple, restricting ~~to~~ operations to $\Omega(\omega)$ produces the sequentially compact spaces and ie $F_p(\omega)$ one gets $\aleph_0$-$\wedge$-semilattices. So maybe restricting to $F(\omega)$ might produce something interesting

§4 Miscellaneous Junk & Stupid questions

(a) Using $E(X) = \{\text{order filters in } 2^X\}$ one gets as E-algebras completely distributive lattices and $\Omega \hookrightarrow E$ as before tells one the known facts about topologizing such.

(b) If we let $\mathbb{T} = (T, \eta, \mu)$ be the compact T_2 $\wedge$-semilattice triple then there must exist natural transformations

$$\begin{tikzcd} & \Omega \arrow[dr, "\alpha"] & & \\ I \arrow[ur, "\eta_\Omega"] \arrow[dr, "\eta_{F_p}"'] & & T \arrow[r, twoheadrightarrow, "\gamma"] & F \\ & F_p \arrow[ur, "\beta"'] & & \end{tikzcd}$$

ident monad

which are ~~triple~~ monad (=triple) morphisms

Therefore T is the push out in the category of triples over Sets (Aghh!) Don't think this helps anything.

(c) As continuous lattices are Monadic over T_0-spaces, what is the correct notion of operation there? Seems to depend on what function-space topology one takes.

References:

(1) Saunders MacLane: Category Theory for the Working Mathematician. GTM 5, Springer Verlag, Chapter VI.

(2) Seminar on Triples and Categorical Homology Theory LNM 80 Springer Verlag. p91 – 118

(3) Day, Filter monads, Cont Lattices and Closure systems Can J. Math 27(1975) 50-59

(4) Dana Scott SCS 23 Aug 76.