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THE VALUE OF INFORMATION UNDER PARTIAL INFORMATION FOR EXPONENTIAL UTILITY

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Abstract. The paper investigates the value of information to an investor under the partial information setting for exponential utility. The only information available to the investor is the one generated by the asset price processes and, in particular, the underlying appreciation rate of the risky asset cannot be observed directly. Filtering theory is used to find a filtered estimate of the underlying appreciation rate. This brings about two maximisation problems from which we determine the optimal expected utilities of wealth under partial and full information, via Hamilton-Jacobi-Bellman equations. The value of information is, therefore, calculated as the difference between the two optimal expected utilities. The effect of parameter changes on the value of information is determined by carrying out numerical simulations.

1. Introduction

The accessibility and flow of information can be associated with a financial value and this value is referred to as the value of information. The value of information which depends, in general, on the whole model, assets, strategies and agents preferences, plays a crucial role in all behavioural sciences, but particularly in finance, see Yang et al. [18]. In particular, Copeland and Friedman [5] used it to make optimal informed decisions, while Gottardi and Rahi [8] used it for portfolio management purposes.

The present paper examines the value of information of an investor by studying the utility maximisation problems from terminal wealth for the cases: partial information and full information. In our case, full information refers to the case when the investor is aware of the appreciation rate of the risky asset, and partial information means that the investor can observe the asset price only but not the appreciation rate and the driving Brownian motion. The case of partial information is more realistic as asset prices are published and are available to the public whereas the appreciation rate and the driving Brownian motion are not.

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The optimisation setting with full information goes back to Merton [15], who solved the problem in a Black-Scholes environment via the Hamiltonian-Jacobi-Bellman (HJB) equation and dynamic programming. Merton's work was extended by several authors, see for example, Karatzas et al. [9], Cox and Huang [6], Kohatsu-Higa and Sulem [10] and Björk et al. [2]. However, the optimal portfolio is characterised through a representation theorem for martingales and explicit solutions may be obtained in very few cases.

Utility maximisation problems under partial information have been studied via the stochastic filtering techniques and by using martingale approach, see for example, Lakner [11, 12], Pham and Quenez [16], Sass and Haussmann [17], Yang and Ma [19]. In addition, dynamic programming approach has been applied to solve utility maximisation problems under various setups, see for example, Yang et al. [18], Brendle [3, 4], Bai and Guo [1], Mania and Santacroce [14]. Lakner [11, 12] studied the problem in general terms and derived, using martingale methods, the structure of the optimal investment and consumption strategies. Explicit expressions for the terminal wealth and the optimal portfolio strategy are derived in the case of logarithm, power, and exponential utilities. However, no numerical solutions were provided. Brendle [3, 4] treated the case where the linear diffusion maybe correlated with the stock prices and obtained explicit results for power and exponential utilities. Sass and Haussmann [17] provided some numerical results but the results are limited to the case where the unobserved process is a finite state Markov process. A few papers addressed the issue of the value of information, see for example, Yang et al. [18], Brendle [3, 4], Yang and Ma [19].

The focus of the present paper is on the value of information which is not the case in most of the cited papers. The special feature is that the only information available to the investor is the one generated by the asset price, and the unobservable process will be modelled by a linear stochastic differential equation. The two level of observations correspond to whether the appreciation rate and the driving Brownian motion are observable or not. We are in the same framework studied in Yang et al. [18] and Bai and Guo [1]. Instead of using the logarithmic utility as in Yang et al. [18], we use the exponential utility. We illustrate how model parameters such as the volatility and the mean risk-aversion parameters affect the value of information. Bai and Guo [1] obtained closed-form solutions in two cases of exponential utility and logarithmic utility. We use stochastic filtering theory to derive analytical tractable formulae for the optimal expected utilities from which we define the value of information. In addition, numerical results are presented. We mention that the value of information is not explicitly computed in Bai and Guo [1] and no numerical results are given.

The main results are contained in Theorem 4.1 and Theorem 5.1. Theorem 4.1 gives the optimal expected utility function and its corresponding optimal trading strategy for the partial information case when the exponential utility function is used. Theorem 4.2 is a verification result which verifies that the results in Theorem 4.1 are indeed optimal for the problem under consideration. Theorem 5.1 gives the optimal expected utility function and its corresponding optimal trading strategy for the full information case when the exponential utility function is used. Theorem 5.2 is a verification result which verifies that the results in Theorem 5.1 are indeed optimal. Definition 6.1 gives the value of information for the exponential utility case.

The paper is structured as follows. In Section 2, we describe the market model, the information structure and formulate the optimization problem. In Section 3 we construct the filtering estimate for the appreciation rate. We also show how wealth process can be reduced to a wealth process with full information via filtering arguments. In Section 4 we derive the optimal expected utility and its corresponding optimal trading strategy under partial information using the dynamic programming approach. A verification theorem is stated and proved. Section 5 deals with the optimal expected utility and its corresponding optimal trading strategy under full information. We also state and prove the verification theorem. In Section 6, we present the definition of the value of information. Numerical results are presented in Figures 1 − 3 in Section 7. The figures show the impact of changes in model parameters on the value of information. Some economical and intuitive explanations are also presented. A brief conclusion is given in Section 8.

2. Model Setup

Suppose $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space equipped with a filtration $\mathbb{F} = \{\mathcal{F}\}_{0\leq t\leq T}$ satisfying the usual conditions, and that $T > 0$ is a fixed time horizon. We consider a financial market with two types of assets which are a risk-free asset (e.g a bond) and a risky asset (e.g a stock). The price at any time t of the risk-free asset is denoted by S_t^0 whereas that of the risky asset is denoted by S_t. The price of the risk-free asset S_t^0 is governed by an equation of the form

$$dS_t^0 = rS_t^0 dt,\ S_0^0 = S^0, \tag{2.1}$$

where the interest rate $r > 0$ is constant. The price of the risky asset $S = (S_t)$ is given by

$$dS_t = S_t(\mu_t dt + \sigma dW_t^1),\ S_0 = S \geq 0 \tag{2.2}$$

where the volatility σ is a known positive constant and W^1 is a Brownian motion. We assume that the volatility σ is non-singular, hence invertible. In contrast to most of the literature on real option theory but similar to Lakner [11], the appreciation rate process is assumed to be a stochastic process and governed by

$$d\mu_t = (\kappa_1\mu_t + \kappa_0)dt + \beta_1 dW_t^1 + \beta_2 dW_t^2,\ 0 \leq t \leq T, \tag{2.3}$$

where κ_0, κ_1, β_1 and β_2 are constants and W^2 is a Brownian motion which is independent of W^1. If κ_1 is negative then the mean appreciation rate process will be an Ornstein-Uhlenbeck process with mean reverting drift.

We denote the total amount of the investor's wealth at time t with X_t and the amount invested into the risky asset with π_t. The remains $X_t - \pi_t$ will be invested into the riskless asset. We refer to π_t as the portfolio strategy. Under the assumption that the portfolio is self-financing, the total wealth process X_t evolves according to the dynamics

$$\begin{aligned} dX_t &= \pi_t \frac{dS_t}{S_t} + (X_t - \pi_t)\frac{dS_t^0}{S_t^0} \\ &= (r_t X_t + \pi_t(\mu_t - r_t))\, dt + \pi_t \sigma dW_t^1. \end{aligned} \tag{2.4}$$

At each time $t < T$, the investor chooses the portfolio position based on information available to them at that time. The investor's information can be modelled by a filtration. We describe two filtrations:

(1) Consider the case where the investor observes the asset price but not the appreciation rate. In this setting the investment strategy should be adapted to the filtration $\{\mathcal{G}_t : t \geq 0\}$ where

$$\mathcal{G}_t = \sigma(S_s, 0 \leq s \leq t).$$

This case will be referred to as the case of partial information.

(2) Consider the case where the information is generated by the noise processes. In this setting the investment strategy should be adapted to the filtration $\{\mathcal{F}_t : t \geq 0\}$ where

$$\mathcal{F}_t = \sigma(W_s^1, W_s^2, 0 \leq s \leq t).$$

This case will be referred to as the case of full information.

We denote by

$$\begin{aligned} \mathcal{A} \;:=\; &\{\pi = (\pi_s)_{t \leq s \leq T} :\ \pi_s \in \mathbb{R},\ \pi \text{ is } \mathcal{G}_s - \text{adapted}, (2.4) \\ &\text{has a unique solution, and } \int_t^T (\pi_s)^2 ds < \infty \text{ a.s.}\Big\} \end{aligned} \tag{2.5}$$

the set of admissible portfolio strategies over the time horizon $[t, T]$. Let $U : \mathbb{R}_+ \to \mathbb{R}$ be an increasing, concave and differentiable utility function. The objective of the investor is to maximise the expected utility from terminal wealth. Precisely, the optimal portfolio problem is formulated as follows:

$$\max_{\pi \in \mathcal{A}} \mathbb{E}\left[U(X_T)\right] \tag{2.6}$$

subject to (2.3) and (2.4).

We note that, since $\{S_t\}_{0 \leq t \leq T}$ is observable, it follows that $\{X_t\}_{0 \leq t \leq T}$ can be observed but $\{\mu_t\}_{0 \leq t \leq T}$ is an unobservable state process. Therefore, the problem to be solved is that of maximising expected utility of terminal wealth over the class of portfolio strategies that are adapted to the observable information $\mathcal{G}$. This leads to an optimal problem under partial information.

In the present paper we consider the optimisation problem (2.6) under exponential utility function:

$$U(x) = 1 - e^{-\eta x}, \tag{2.7}$$

where $\eta > 0$ is the risk aversion parameter and x is the investor's initial wealth. The additive term 1 in (2.7) keeps the range of the function between zero and one over the domain of non-negative values for x. The exponential utility function is one of the most used utilities to represent investor's attitude towards risk in portfolio optimisation. It has a constant absolute risk aversion which means that the investor has the same risk preferences for random outcomes independent of his wealth.

3. Filtering Estimation for the Appreciation Rate

In this section we apply the filtering technique to estimate the appreciation rate $\{\mu_t\}_{0\leq t\leq T}$ given the information flow $\mathcal{G}_t$. The uncertainty of the partially informed investor over μ_t is described by a normal distribution with mean

$$m_t = \mathbb{E}\left[\mu_t|\mathcal{G}_t\right], \tag{3.1}$$

and variance

$$\gamma_t = \mathbb{E}\left[\left(\mu_t - m_t\right)^2|\mathcal{G}_t\right] \tag{3.2}$$

with the convention that $m_0 = \mathbb{E}[\mu_0]$ and $\gamma_0 = Var(\mu_0)$.

Then, as the system (2.3) and (2.4) is conditionally Gaussian, the conditional law of μ_t with respect to $\mathcal{G}_t$ is also of the Gaussian type with mean m_t and variance γ_t and the pair (m,γ) satisfies a system of linear equations given by the so-called Kalman filter (see Brendle [4]). Results from Liptser and Shiryaev [13] show that the mean m_t follow the stochastic differential equation

$$dm_t = (\kappa_1 m_t + \kappa_0)dt + \frac{\beta_1\sigma_t + \gamma_t}{\sigma^2}\left(\frac{dS_t}{S_t} - m_t dt\right). \tag{3.3}$$

Moreover, the variance γ_t satisfies the Riccati equation

$$\frac{d\gamma_t}{dt} = 2\kappa_1\gamma_t + \beta_1^2 + \beta_2^2 - \left(\frac{\beta_1\sigma_t + \gamma}{\sigma_t}\right)^2. \tag{3.4}$$

We note that (3.4) is a deterministic equation on γ_t.

As in Liptser and Shiryaev [13], we define the innovation process:

$$d\overline{W} := \frac{1}{\sigma}\left(\frac{dS_t}{S_t} - m_t dt\right), \tag{3.5}$$

where $\overline{W} = \{\overline{W_t}\}_{0\leq t\leq T}$ is a Brownian motion with respect to the probability space $(\Omega, \mathcal{G}_T, \{\mathcal{G}_t\}_{0\leq t\leq T}, \mathbb{P})$ for t $\in [0,T]$. Combining (2.4) and (3.5) we obtain

$$dX_t = \left(r_t X_t + \pi_t(m_t - r_t)\right)dt + \pi_t\sigma d\overline{W}. \tag{3.6}$$

Similarly, (3.3) can be written as

$$dm_t = (\kappa_1 m_t + \kappa_0)dt + \frac{\beta_1\sigma + \gamma_t}{\sigma}d\overline{W}. \tag{3.7}$$

We note that (3.6) and (3.7) describe the dynamics of the wealth process from the perspective of a partially informed agent.

4. Optimal Investment under Partial Information

The investor wants to maximise the expected utility of the wealth at some future time $T > 0$ under partial information for exponential utility. Suppose that at time $t = 0$ we have $X_0 = x > 0$. Moreover, we assume that the utility function is given by (2.7). The original problem reduces to maximising (2.6) subject to constraints (3.6) and (3.7), that is, under partial information, the value function for the investor in the reduced model is then

$$\Phi(t,x,m) = \sup_{\pi\in\mathcal{G}} \mathbb{E}^{t,x,m}[U(X_T)], \tag{4.1}$$

where $\mathbb{E}^{t,x,m}[\cdot]$ is the conditional expectation given that $X_t = x$ and $m_t = m$. Let $\mathcal{C}^{1,2,2}$ denote the functional space of the function $\Phi(t,x,m)$ such that $\Phi(t,x,m)$ and its partial derivatives $\Phi_t,\ \Phi_x,\ \Phi_{xx},\ \Phi_m,\ \Phi_{xm},\ \Phi_{mm}$ are continuous on $[0,T]\times\mathbb{R}^+\times\mathbb{R}$. Now we can define a differential operator L^{π_t}:

$$\begin{aligned}
\left(L^{\pi_t}\Phi\right)(t,x,m) &= \Phi_t + (\kappa_1 m + \kappa_0)\Phi_m + (r_t x + (m-r_t)\pi_t)\Phi_x \\
&\quad + \frac{1}{2}\left(\frac{\beta_1\sigma+\gamma_t}{\sigma}\right)^2\Phi_{mm} + \frac{1}{2}\sigma^2\pi_t^2\Phi_{xx} + (\beta_1\sigma+\gamma_t)\pi_t\Phi_{xm}.
\end{aligned}$$

By using the principle of stochastic optimality, we have the following HJB equation:

$$\sup_{\pi\in\mathcal{G}}\left\{\left(L^{\pi_t}\Phi\right)(t,x,m)\right\} = 0, \tag{4.2}$$

with the terminal condition

$$\Phi(T,x,m) = U(x), \tag{4.3}$$

for $(x,m)\in\mathbb{R}^+\times\mathbb{R}$. Therefore for each (t,x,m) we try to find the value π_t^* which maximises (4.2). We now assume that $H(t,x,m)\in\mathcal{C}^{1,2,2}$ is a candidate solution of the HJB equation (4.2), that is,

$$\sup_{\pi\in\mathcal{G}}\left\{\left(L^{\pi_t}H\right)(t,x,m)\right\} = 0,\ \ H(T,x,m) = U(x). \tag{4.4}$$

From (4.4) we find out that the first order condition for an optimal investment strategy is given by

$$\pi_t^* = -\frac{(m-r_t)H_x + (\beta_1\sigma+\gamma_t)H_{xm}}{\sigma^2 H_{xx}}. \tag{4.5}$$

Substituting (4.5) into (4.4) we get the following nonlinear boundary value problem for H:

$$\begin{aligned}
&H_t + (\kappa_1 m+\kappa_0)H_m + r_t x H_x + \frac{1}{2}\left(\frac{\beta_1\sigma+\gamma_t}{\sigma}\right)^2 H_{mm} \\
&-\frac{\left((m-r_t)H_x + (\beta_1\sigma+\gamma_t)H_{xm}\right)^2}{2\sigma^2 H_{xx}} = 0
\end{aligned} \tag{4.6}$$

with terminal condition

$$H(T,x,m) = 1 - e^{-\eta x}. \tag{4.7}$$

To find the solution $H(t,x,m)$ we proceed as follows. We look for a candidate solution of (4.6) in the form:

$$H(t,x,m) = 1 - e^{-\eta x e^{r(T-t)} + f(t,m)}. \tag{4.8}$$

Substituting (4.8) and its partial derivatives into (4.6), we obtain:

$$\dot{f}(t,m) + (\kappa_1 m + \kappa_0) f'(t,m) + \frac{1}{2}\left(\frac{\beta_1\sigma + \gamma_t}{\sigma}^2\right)(f'^2(t,m) + f''(t,m)) \\ -\frac{((m - r_t) + (\beta_1\sigma + \gamma_t) f'(t,m))^2}{2\sigma^2} = 0. \tag{4.9}$$

We assume the solution of the form

$$f(t,m) = A(t)m^2 + B(t)m + C(t). \tag{4.10}$$

Differentiation yields

$$\dot{f}(t,m) = A'(t)m^2 + B'(t)m + C'(t). \tag{4.11}$$

$$f'(t,m) = 2mA(t) + B(t), \tag{4.12}$$

$$f''(t,m) = 2A(t). \tag{4.13}$$

Substituting (4.11) - (4.13) into (4.9) we obtain:

$$[A'(t)m^2 + B'(t)m + C'(t)] + (\kappa_1 m + \kappa_0)[2mA(t) + B(t)] \\ +\frac{1}{2}\left(\frac{\beta_1\sigma + \gamma_t}{\sigma}\right)^2 [2A(t) + (2mA(t) + B(t))^2] \\ -\frac{((m - r_t) + (\beta_1\sigma + \gamma_t)[2mA(t) + B(t)])^2}{2\sigma^2} = 0. \tag{4.14}$$

Rearranging and grouping terms according to the order of m we obtain

$$\left[A'(t) + 2\kappa_1 A(t) + 2\left(\frac{\beta_1\sigma + \gamma_t}{\sigma}\right)^2 A^2(t) - \frac{(1 + 2(\beta_1\sigma + \gamma_t)A(t))^2}{2\sigma^2}\right] m^2 \\ + \left[B'(t) + \kappa_1 B(t) + 2\kappa_0 A(t) + 2\left(\frac{\beta_1\sigma + \gamma_t}{\sigma}\right)^2 A(t)B(t) \right. \\ \left. -\frac{(1 + 2(\beta_1\sigma + \gamma_t)A(t))((\beta_1\sigma + \gamma_t)B(t) - r_t)}{\sigma^2}\right] m \\ + \left[C'(t) + \kappa_0 B(t) + \frac{1}{2}\left(\frac{\beta_1\sigma + \gamma_t}{\sigma}\right)^2 (2A(t) + B^2(t)) \right. \\ \left. - \frac{((\beta_1\sigma + \gamma_t)B(t) - r_t)^2}{2\sigma^2}\right] = 0. \tag{4.15}$$

Equation (4.15) can only be identical to zero if the three brackets are identical to zero. Using this fact we obtain the following three ordinary differential equations for the functions $A(t)$, $B(t)$ and $C(t)$:

$$A'(t) = \frac{(1 + 2(\beta_1\sigma + \gamma_t)A(t))^2}{2\sigma^2} - 2\left(\kappa_1 + \left(\frac{\beta_1\sigma + \gamma_t}{\sigma}\right)^2 A(t)\right) A(t) \tag{4.16}$$

$$
\begin{aligned}
B'(t) &= \frac{(1+2(\beta_1\sigma+\gamma_t)A(t))((\beta_1\sigma+\gamma_t)B(t)-r_t)}{\sigma^2} \\
&\quad - \left(\kappa_1 + 2\left(\frac{\beta_1\sigma+\gamma_t}{\sigma}\right)^2 A(t)\right)B(t) - 2\kappa_0 A(t) \qquad (4.17)
\end{aligned}
$$

$$
C'(t) = \frac{((\beta_1\sigma+\gamma_t)B(t)-r_t)^2}{2\sigma^2} - \frac{1}{2}\left(\frac{\beta_1\sigma+\gamma_t}{\sigma}\right)^2 (2A(t)+B^2(t)) - \kappa_0 B(t) \quad (4.18)
$$

with $A(T) = B(T) = C(T) = 0$. We therefore, have the following result.

Theorem 4.1. *Suppose the utility function is given by (2.7). Then the optimal expected utility for the problem (4.4) is given by*

$$
H(t,x,m) = 1 - e^{-\eta x e^{r(T-t)} + f(t,m)}, \qquad (4.19)
$$

where $A(t)$, $B(t)$ and $C(t)$ are determined by (4.16), (4.17) and (4.18), respectively. The corresponding optimal trading strategy π_t^* is given as:

$$
\pi_t^* = \frac{m - r_t + (\beta_1\sigma+\gamma_t)(2A(t)m + B(t))}{\eta\sigma^2 e^{r(T-t)}}. \qquad (4.20)
$$

The following verification result verifies that the results in Theorem 4.1 are optimal for the problem (4.1).

Theorem 4.2. *Suppose $H^*(t,x,m) \in \mathcal{C}^{1,2,2}$ given by (4.19) is a candidate solution of the HJB equation (4.4), that is, $H^*(t,x,m)$ satisfies*

$$
\sup_{\pi\in\mathcal{G}}\{L^{\pi_t}H(t,x,m)\} = 0, \;\; H(T,x,m) = 1 - e^{-\eta x}. \qquad (4.21)
$$

Then, $\Phi(t,x,m) \le H^*(t,x,m)$ for any admissible strategy $\pi_t \in \mathcal{G}$. In addition, if there exists an optimal portfolio $\pi_t^* \in \mathcal{G}$ such that

$$
\pi_t^* \in \arg\sup_{\pi\in\mathcal{G}}\{L^{\pi_t}\Phi(t,x,m)\},
$$

then, when $\pi_t = \pi_t^*$, $\Phi(t,x,m) = H^*(t,x,m)$.

Proof. Suppose $\mathcal{O} = [0,\infty)\times[0,\infty)$, we choose a sequence of bounded open sets $\mathcal{O}$ satisfying $\mathcal{O}_i \subset \mathcal{O}_{i+1} \subset \mathcal{O}$, $i = 1,2,...$ and $\mathcal{O} = \bigcup_{i=1}^{\infty}\mathcal{O}_i$. For all $(x,m)\in\mathcal{O}$, assume that the exit time of (x_t,m_t) from $\mathcal{O}$ is denoted by τ_i, when $i\to\infty$ we obtain $\tau_i \wedge T \to T$.

We consider an arbitrary strategy $\pi_t \in \mathcal{G}$. An application of Itô's formula for $H(t,x,m)$ on $[t,T]$ yields

$$
\begin{aligned}
H(T,x_T,m_T) &= H(t,x,m) + \int_t^T L^{\pi_t}H(s,x_s,m_s)ds \\
&\quad + \int_t^T \sigma\pi_t H_x(s,x_s,m_s)d\overline{W}_s \\
&\quad + \int_t^T \frac{\beta_1\sigma+\gamma_t}{\sigma}H_m(s,x_s,m_s)d\overline{W}_s.
\end{aligned}
$$

We note that $\sup_{\pi\in\mathcal{G}}\{L^{\pi_t}H(t,x,m)\} = 0$. This implies that the variational inequality $L^{\pi_t}H(t,x,m) \leq 0$. Therefore we have

$$\begin{aligned} H(T,x_T,m_T) \quad &\leq \quad H(t,x,m) + \int_t^T \sigma\pi_t H_x(s,x_s,m_s)d\overline{W}_s \\ &\qquad + \int_t^T \frac{\beta_1\sigma+\gamma_t}{\sigma} H_m(s,x_s,m_s)d\overline{W}_s. \end{aligned} \tag{4.22}$$

The last two terms on the right hand side of (4.22) are square-integrable martingales whose expectation is zero. Hence, we have

$$\mathbb{E}^{t,x,m}[H(T,x_T,m_T)] \leq H(t,x,m).$$

Furthermore, taking the supremum, we get

$$\sup_{\pi\in\mathcal{G}} \mathbb{E}^{t,x,m}[H(T,x_T,m_T)] \leq H(t,x,m).$$

Recalling the definition of $\Phi(t,x,m)$, we have

$$\Phi(t,x,m) \leq H(t,x,m),$$

which implies that

$$\Phi(t,x,m) \leq H^*(t,x,m),$$

Next we suppose that

$$\mathbb{E}[H(\tau_{i\wedge T}, x_{\tau_i\wedge T}, m_{\tau_i\wedge T})] \leq \infty$$

for some specific strategy $\pi_t^* \in \mathcal{G}$. Applying Itô's formula to $H(t,x,m)$ on $[0,\tau_i\wedge T]$ once again, we obtain

$$\begin{aligned} H(\tau_i\wedge T, x_{\tau_i\wedge T}, m_{\tau_i\wedge T}) \quad &= \quad H(0,x_0,m_0) + \int_0^{\tau_i\wedge T} L^{\pi_t}H(s,x_s,m_s)ds \\ &\qquad + \int_0^{\tau_i\wedge T} \sigma\pi_t H_x(s,x_s,m_s)d\overline{W}_s \\ &\qquad + \int_0^{\tau_i\wedge T} \frac{\beta_1\sigma+\gamma_t}{\sigma} H_m(s,x_s,m_s)d\overline{W}_s. \end{aligned} \tag{4.23}$$

For some specific strategy satisfying (4.1), that is, $L^{\pi_t}H(s,x_s,m_s) = 0$ and the last two terms of (4.23) are also square-integrable martingales. Hence, taking the expectation on both sides of (4.23), we obtain

$$\mathbb{E}[H(\tau_i\wedge T, x_{\tau_i\wedge T}, m_{\tau_i\wedge T}) = H(0,x_0,m_0) < \infty. \tag{4.24}$$

Now we set $\Phi(t,x,m) = H(t,x,m)$ for the optimal strategy π_t^*. Using Itô's formula on $H(t,x,m)$ on $[t,\tau_i\wedge T]$ once more, similarly, we obtain

$$\begin{aligned} H(\tau_i\wedge T, x_{\tau_i\wedge T}, m_{\tau_i\wedge T}) \quad &= \quad H(t,x,m) + \int_t^{\tau_i\wedge T} L^{\pi_t}H(s,x_s,m_s)ds \\ &\qquad + \int_t^{\tau_i\wedge T} \sigma\pi_t H_x(s,x_s,m_s)d\overline{W}_s \\ &\qquad + \int_t^{\tau_i\wedge T} \frac{\beta_1\sigma+\gamma_t}{\sigma} H_m(s,x_s,m_s)d\overline{W}_s. \end{aligned} \tag{4.25}$$

Taking the conditional expectation, we have

$$H(t,x,m)=\mathbb{E}^{t,x,m}[H(\tau_i\wedge T, x_{\tau_i\wedge T}, m_{\tau_i\wedge T})].$$

Taking the limit on both sides we have

$$H(t,x,m)=\lim_{i\to\infty}\mathbb{E}^{t,x,m}[H(\tau_i\wedge T, x_{\tau_i\wedge T}, m_{\tau_i\wedge T}).$$

In addition, we have

$$\begin{aligned}\Phi(t,x,m) &= \sup_{\pi\in\mathcal{G}}\mathbb{E}^{t,x,m}[U(X_T)\\ &= \lim_{i\to\infty}\mathbb{E}^{t,x,m}[H(\tau_i\wedge T, x_{\tau_i\wedge T}, m_{\tau_i\wedge T})]\\ &= H(t,x,m)\end{aligned}$$

which means that $\pi_t=\pi_t^*$ and we have $\Phi(t,x,m)=H^*(t,x,m)$, and this completes the proof. $\square$

5. Optimal Investment under Full Information

Consider the wealth dynamics provided by (2.3) and (2.4). Suppose that, starting with wealth $X_0=x>0$ at time $t=0$, the investor wants to maximise the expected utility of the wealth at some future time $T>0$ under full information for exponential utility. The utility function is given by (2.7) subject to the constraints (2.3) and (2.4), that is, under full information, the value function for the investor is then

$$\Phi(t,x,\mu)=\sup_{\pi\in\mathcal{F}}\mathbb{E}^{t,x,\mu}[U(X_T)], \tag{5.1}$$

where $\mathbb{E}^{t,x\mu}[\cdot]$ is the conditional expectation given that $X_t=x$ and $\mu_t=\mu$. As in the previous section, we let $\mathcal{C}^{1,2,2}$ denote the functional space of the function $\Phi(t,x,\mu)$ such that $\Phi(t,x,\mu)$ and its partial derivatives $\Phi_t,\ \Phi_x,\ \Phi_{xx},\ \Phi_\mu,\ \Phi_{x\mu},\ \Phi_{\mu\mu}$ are continuous on $[0,T]\times\mathbb{R}^+\times\mathbb{R}$. Now we can define a differential operator L^{π_t}:

$$\begin{aligned}\left(L^{\pi_t}\Phi\right)(t,x,\mu) &= \Phi_t+(r_tx+\pi_t(\mu-r_t))\Phi_x+\frac{1}{2}\sigma^2\pi_t^2\Phi_{xx}+(\kappa_1\mu+\kappa_0)\Phi_\mu\\ &\quad+\frac{1}{2}(\beta_1^2+\beta_2^2)\Phi_{\mu\mu}+\beta_1\sigma\pi_t\Phi_{x\mu}.\end{aligned} \tag{5.2}$$

By using the principle of stochastic optimality, we have the following HJB equation:

$$\sup_{\pi\in\mathcal{F}}\left\{\left(L^{\pi_t}\Phi\right)(t,x,\mu)\right\}=0, \tag{5.3}$$

with terminal condition

$$\Phi(T,x,\mu)=U(x), \tag{5.4}$$

for $(x,\mu)\in\mathbb{R}^+\times\mathbb{R}$. Therefore for each (t,x,μ) we try to find the value π_t^* which maximises (5.3). As in the previous section, we now assume that $H(t,x,\mu)\in\mathcal{C}^{1,2,2}$ is a candidate solution of the HJB equation (5.3), that is,

$$\sup_{\pi\in\mathcal{F}}\left\{\left(L^{\pi_t}H\right)(t,x,\mu)\right\}=0,\ \ H(T,x,\mu)=U(x). \tag{5.5}$$

From (5.5) we find out that the first order condition for an optimal investment strategy is given by

$$\pi_t^* = -\frac{(\mu - r_t)H_x + \beta_1 \sigma H_{x\mu}}{\sigma^2 H_{xx}}. \tag{5.6}$$

Substituting (5.6) into (5.3) we get the following nonlinear boundary value problem for H:

$$H_t + r_t x H_x + (\kappa_1 \mu + \kappa_0) H_\mu + \frac{1}{2}\left(\beta_1^2 + \beta_2^2\right) H_{\mu\mu} - \frac{\left((\mu - r_t)H_x + \beta_1 \sigma H_{x\mu}\right)^2}{2\sigma^2 H_{xx}} = 0. \tag{5.7}$$

with terminal condition

$$H(T, x, \mu) = 1 - e^{-\eta x}. \tag{5.8}$$

To find the solution $H(t, x, m)$ we proceed as follows. We look for a candidate solution of (5.7) of the form:

$$H(t, x, \mu) = 1 - e^{-\eta x e^{r(T-t)} + h(t,m)}. \tag{5.9}$$

Substituting (5.9) into (5.7), we obtain

$$\begin{aligned} &\dot{h}(t,\mu) + (\kappa_1 \mu + \kappa_0) h'(t,\mu) + \frac{1}{2}\left(\beta_1^2 + \beta_2^2\right)\left({h'}^2(t,\mu) + h''(t,\mu)\right) \\ &- \frac{\left[(\mu - r_t) + \beta_1 \sigma h'(t,\mu)\right]^2}{2\sigma^2} = 0. \end{aligned} \tag{5.10}$$

We assume a solution of the form

$$h(t,\mu) = D(t)\mu^2 + E(t)\mu + F(t). \tag{5.11}$$

Differentiation yields

$$\dot{h}(t,\mu) = D'(t)\mu^2 + E'(t)\mu + F'(t), \tag{5.12}$$

$$h'(t,\mu) = 2\mu D(t) + E(t), \tag{5.13}$$

$$h''(t,\mu) = 2D(t). \tag{5.14}$$

Substituting (5.12) - (5.14) into (5.10) we obtain:

$$\begin{aligned} &[D'(t)\mu^2 + E'(t)\mu + F'(t)] + (\kappa_1 \mu + \kappa_0)[2D(t)\mu + E(t)] \\ &+ \frac{1}{2}\left(\beta_1^2 + \beta_2^2\right)\left[2D(t) + (2D(t)\mu + E(t))^2\right] \\ &- \frac{\left[(\mu - r_t) + \beta_1 \sigma [2D(t)\mu + E(t)]\right]^2}{2\sigma^2} = 0. \end{aligned} \tag{5.15}$$

Rearranging and grouping the terms according to the order of μ we obtain

$$\begin{aligned}
&\left[D'(t) + 2\kappa_1 D(t) + 2(\beta_1^2+\beta_2^2)D^2(t) - \frac{(1+2\beta_1\sigma D(t))^2}{2\sigma^2}\right]\mu^2 \\
&+ \big[E'(t) + \kappa_1 E(t) + 2\kappa_0 D(t) + 2(\beta_1^2+\beta_2^2)D(t)E(t) \\
&- \frac{(1+2\beta_1\sigma D(t))(\beta_1\sigma E(t) - r_t)}{\sigma^2}\bigg]\mu \\
&+ \bigg[F'(t) + \kappa_0 E(t) + (\beta_1^2+\beta_2^2)D(t) + \frac{1}{2}(\beta_1^2+\beta_2^2)E^2(t) \\
&- \frac{(\beta_1\sigma E(t) - r_t)^2}{2\sigma^2}\bigg] = 0.
\end{aligned} \tag{5.16}$$

Equation (5.16) can only be identical to zero if the three brackets are identical to zero. Using this fact we obtain the following three ordinary differential equations for the functions $D(t)$, $E(t)$ and $F(t)$:

$$D'(t) = \frac{(1+2\beta_1\sigma D(t))^2}{2\sigma^2} - 2\kappa_1 D(t) - 2(\beta_1^2+\beta_2^2)D^2(t), \tag{5.17}$$

$$\begin{aligned}
E'(t) \quad = \quad & \frac{(1+2\beta_1\sigma D(t))(\beta_1\sigma E(t) - r_t)}{\sigma^2} - \kappa_1 E(t) - 2\kappa_0 D(t) - 2(\beta_1^2 \\
& +\beta_2^2)D(t)E(t),
\end{aligned} \tag{5.18}$$

$$F'(t) = \frac{(\beta_1\sigma E(t) - r_t)^2}{2\sigma^2} - \kappa_0 E(t) - (\beta_1^2+\beta_2^2)D(t) - \frac{1}{2}(\beta_1^2+\beta_2^2)E^2(t), \tag{5.19}$$

where $D(T) = E(T) = F(T) = 0$. We therefore, have the following result:

Theorem 5.1. *Suppose the utility function is given by (2.7). Then the optimal expected utility for the problem (5.1) is given by*

$$H(t,x,\mu) = 1 - e^{-\eta x e^{r(T-t)} + h(t,\mu)}, \tag{5.20}$$

where $D(t)$, $E(t)$ and $F(t)$ are determined by (5.17), (5.18) and (5.19), respectively. The corresponding optimal trading strategy π_t^* is given as:

$$\pi_t^* = \frac{\mu - r_t + \beta_1\sigma(2D(t)\mu + E(t))}{\eta\sigma^2 e^{r(T-t)}}. \tag{5.21}$$

The following verification result verifies that the results in Theorem 5.1 are optimal for the problem (5.1).

Theorem 5.2. *Suppose $H^*(t,x,\mu) \in \mathcal{C}^{1,2,2}$ given by (5.20) is a candidate solution of the HJB equation (5.5), that is, $H^*(t,x,\mu)$ satisfies*

$$\sup_{\pi\in\mathcal{F}}\{L^{\pi_t}H(t,x,\mu)\} = 0, \;\; H(T,x,\mu) = 1 - e^{-\eta x}. \tag{5.22}$$

Then $\Phi(t,x,\mu) \le H^*(t,x,\mu)$ for any admissible strategy $\pi_t \in \mathcal{F}$. In addition, if there exists an optimal portfolio $\pi_t^* \in \mathcal{F}$ such that

$$\pi_t^* \in \arg\sup_{\pi\in\mathcal{F}}\{L^{\pi_t}\Phi(t,x,\mu)\},$$

then, when $\pi_t = \pi_t^*$, $\Phi(t,x,\mu) = H^*(t,x,\mu)$.

Proof. The proof follow the same arguments as in Theorem 4.2. □

Remark 5.3. The optimal trading strategies in Theorem 4.1 and Theorem 5.1 do not depend on the investor's initial wealth x. The investor makes a decision by considering the state of the market only.

6. The Value of Information

The value of information is therefore calculated as the difference between the two optimal expected utilities under full information and partial information. We note that the initial condition m_0 under partial information case is deterministic whereas the initial condition μ_0 under the full information case is a normal distributed random variable, see [18]. The initial value μ_0 is, however, known to the investor at the outset so, μ_0 can be considered to be a known constant. Under this scenario, the mean appreciation rate process and the wealth process are independent of time and so the optimal expected utility is determined by the present utility value and do not depend on time. Therefore, for the investor with full information the optimal expected utility is given by $\mathbb{E}[\Phi(0, x, \mu_0)]$ where the expectation is taken over μ_0. Since the form of the value function Φ in (5.20) and the density function of the normal distribution are known, it is possible to evaluate $\mathbb{E}[\Phi(0, x, \mu_0)]$. However, under partial information the optimal expected utility is given by $\Phi(t, x, m_0)$, that is, the optimal expected utility depends on m_0 and γ_0 but not on μ_0. The two optimal expected utilities are comparable. Hence, we state the following definition.

Definition 6.1. The value of information, denoted by V, for the exponential utility case is given by

$$V := \mathbb{E}[\Phi(0, x, \mu_0)] - \Phi(t, x, m_0) = \int_{-\infty}^{\infty} \Phi(0, x, \mu_0)\varphi(u)du - \Phi(t, x, m_0), \quad (6.1)$$

where $\varphi(\cdot)$ is the normal probability density function with mean m_0 and variance γ_0.

7. Numerical Results

In this section we show how model parameter changes affect the value of information. We will set our baseline parameters as follows: $r = 0.08$, $\sigma \in [0.2, 0.5]$, $\kappa_0 = 0.1$, $\kappa_1 = 0.2$, $\beta_1 = 0.3$, $\beta_2 = 0.1$, $\gamma_0 \in [0, 0.2]$, $\eta_0 \in [0.1, 0.5]$, $T = 1$, $m_0 = 0.2$ and $x = 0.4$. We have chosen these baseline parameters so that we can compare our results with those in literature, see, for example, [18].

Figure 1 illustrates the impact of changes in the parameter σ when the other parameters are kept constant. The figure shows that the value of information decreases with increase in the volatility parameter σ. This is because a high volatility leads to a high risk which decreases the value of information to a risk-averse investor.

Figure 2 illustrates the impact of changes in the parameter γ when the other parameters are kept constant. The figure shows that the value of information increases with increase in the value of γ_0, that is, the uncertainty of the mean appreciation rate of the wealth value. We also note that when $\gamma_0 = 0$ the value

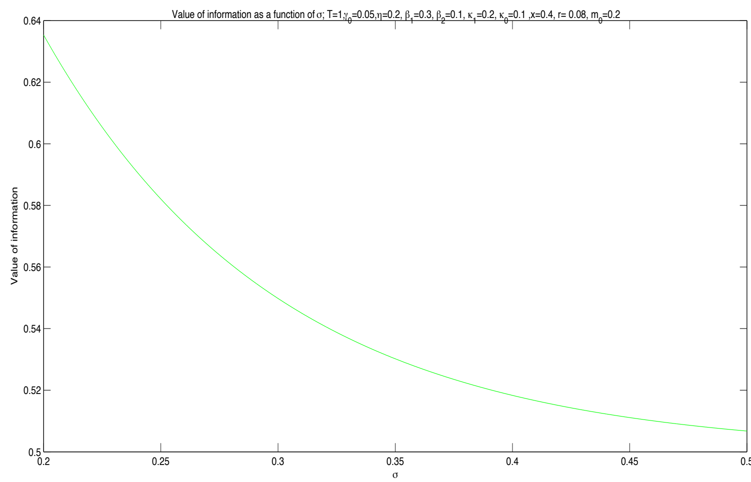

FIGURE 1. The effect of σ on the value of information

of information is zero. This agrees with intuition, see Yang and Ma [19]. We note that γ_0 is the initial uncertainty in μ_0.

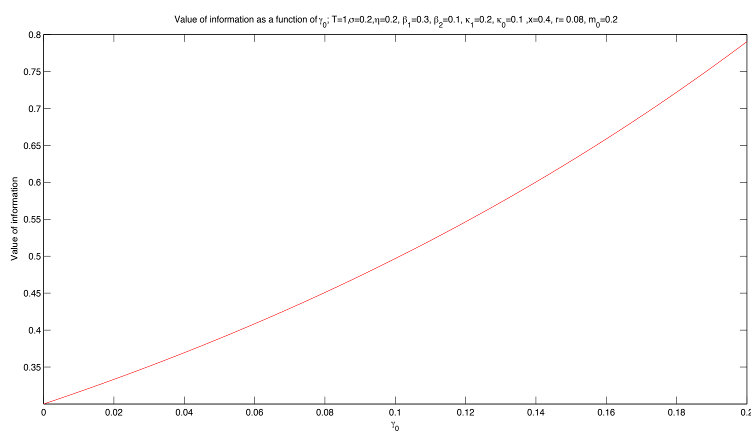

FIGURE 2. The effect of γ_0 on the value of information

Figure 3 shows the value of information as a function of the risk aversion parameter η. For low value of the risk aversion, we have low value of information. The value of information increases with the increase in the risk aversion parameter. This is expected in practice as more risk-averse investors value information more than less risk-averse investors. This intuition is also numerically confirmed in Yang et al. [18] and Ewald et al. [7].

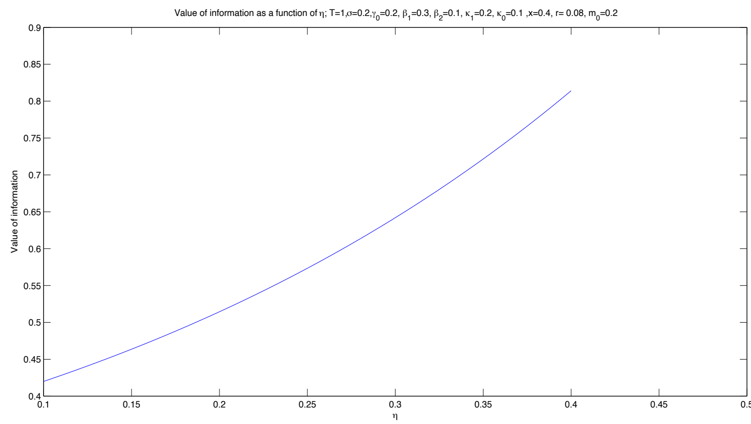

FIGURE 3. The effect of η on the value of information

8. Conclusion

This paper dealt with utility maximisation problems from terminal wealth for the two cases, partial information and full information. The investor could only observe the asset price processes but could not observe the appreciation rate. For exponential utility function, the optimal expected utility and the optimal trading strategy are related to the solution of a semi-linear partial differential equation. Explicit formulae were obtained for the optimal expected utilities and optimal trading strategies in both cases. The formulae indicate that the optimal trading strategies are independent of the initial wealth. The value of information has been defined as the difference between the two optimal expected utilities. The term κ_0 in the unobservable drift term does not have any influence on the value of information. This is because this parameter is not multiplied by a variable that produces uncertainty. Numerical results which show how model parameters affect the value of information were presented. In particular, the value of information decreases with the increase in volatility and increases with increase in both the parameter γ_0 and the risk aversion parameter η.

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