Communications on Stochastic Analysis

Volume 1 | Number 2 Article 6

8-1-2007

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Breton, Jean-Christophe and Privault, Nicolas (2007) "Convex comparison inequalities for exponential jump-diffusion processes," *Communications on Stochastic Analysis*: Vol. 1 : No. 2 , Article 6.
DOI: 10.31390/cosa.1.2.06
Available at: https://digitalcommons.lsu.edu/cosa/vol1/iss2/6

Communications on Stochastic Analysis
Vol. 1, No. 2 (2007) 263-277

CONVEX COMPARISON INEQUALITIES FOR EXPONENTIAL JUMP-DIFFUSION PROCESSES

JEAN-CHRISTOPHE BRETON AND NICOLAS PRIVAULT

Abstract. Given $(M_t)_{t\in\mathbb{R}_+}$ and $(M^*_t)_{t\in\mathbb{R}_+}$ respectively a forward and a backward exponential martingale with jumps and a continuous part, we prove that $E[\phi(M_tM^*_t)]$ is non-increasing in t when ϕ is a convex function, provided the local characteristics of the stochastic logarithms of $(M_t)_{t\in\mathbb{R}_+}$ and of $(M^*_t)_{t\in\mathbb{R}_+}$ satisfy some comparison inequalities. As an application, we deduce bounds on option prices in markets with jumps, in which the underlying processes need not be Markovian. In this setting the classical propagation of convexity assumption for Markov semigroups [4] is not needed.

1. Introduction

Bounds on option prices with convex payoff functions have been obtained by several authors. Theorem 6.2 of [4], for example, states that

$$E[\phi(S_T) \mid S_0 = x] \le E[\phi(S^*_T) \mid S^*_0 = x], \qquad x > 0, \tag{1.1}$$

for any convex function ϕ, provided S and S^* are price processes of the form

$$\frac{dS_t}{S_t} = r_t dt + \sigma_t dW_t$$

and

$$\frac{dS^*_t}{S^*_t} = r_t dt + \sigma^*(t, S^*_t) dW_t,$$

where $(W_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion, under the condition

$$|\sigma_t| \le |\sigma^*(t, S_t)|, \qquad t \in \mathbb{R}_+.$$

The proof of (1.1) relies on the backward Kolmogorov equation, provided the Markov semigroup of $(S^*_t)_{t\in\mathbb{R}_+}$ propagates convexity. A first extension of this type of bound to the jump-diffusion case can be found in [1], and more general results have been later proved in [2] under refined conditions, still under the propagation of convexity hypothesis. Note however that the propagation of convexity property is not always satisfied, even in the (Markovian) jump-diffusion case, cf. e.g. Theorem 4.4 in [3].

In this paper we prove a convex comparison inequality of the form

$$E[\phi(M_tM^*_t)] \le E[\phi(M_sM^*_s)], \qquad 0 \le s \le t, \tag{1.2}$$

2000 *Mathematics Subject Classification.* Primary 60E15; Secondary 60H05, ,60G44, 60G55.
Key words and phrases. Convex concentration, forward-backward stochastic calculus, exponential jump diffusion processes, option prices.

where M_t, M_t^* are respectively forward and backward exponential martingales with jumps and continuous parts, satisfying some conditions. More precisely, (1.2) will hold for convex function $\phi : \mathbb{R} \to \mathbb{R}$, provided the local characteristics of the stochastic logarithms of $(M_t)_{t\in\mathbb{R}_+}$ and of $(M_t^*)_{t\in\mathbb{R}_+}$ satisfy the comparison inequalities assumed in Theorem 3.1 below. Our proof relies on arguments of [6], with the difference that we consider products instead of sums of forward and backward martingales. Moreover the results of [6] require a.s. uniform bounds on the diffusion coefficients which cannot be satisfied for stochastic exponentials. In some results we assume in addition that ϕ' is convex, a condition that can be realized in applications when ϕ is e.g. an exponential payoff function.

If further $E[M_t^*|\mathcal{F}_t^M] = 1$, $t \in \mathbb{R}_+$, where $(\mathcal{F}_t^M)_{t\in\mathbb{R}_+}$ denotes the filtration generated by $(M_t)_{t\in\mathbb{R}_+}$, then Jensen's inequality yields

$$\begin{aligned} E\left[\phi(M_t)\right] &= E\left[\phi\left(M_t E[M_t^*|\mathcal{F}_t^M]\right)\right] \\ &= E\left[\phi\left(E[M_t M_t^*|\mathcal{F}_t^M]\right)\right] \\ &\leq E\left[E\left[\phi(M_t M_t^*)|\mathcal{F}_t^M\right]\right] \\ &= E\left[\phi(M_t M_t^*)\right] \\ &\leq E\left[\phi\left(M_s M_s^*\right)\right], \qquad 0 \leq s \leq t, \end{aligned}$$

and in particular

$$E\left[\phi(M_t)\right] \leq E\left[\phi\left(M_0 M_0^*\right)\right], \qquad t \geq 0. \tag{1.3}$$

We prove (1.2) using forward-backward stochastic calculus, assuming only the convexity of ϕ, and without propagation of convexity, cf. Theorem 3.1.

We note that (1.3) can be read as a bound on option prices, where ϕ is a convex payoff function and M_t is the price of an underlying asset. More precisely, cf. Corollaries 4.2, 5.1 and 6.2, it yields bounds of the form

$$E[\phi(S_T) \mid S_0 = x] \leq E[\phi(S_T^*) \mid S_0^* = x], \qquad x > 0, \tag{1.4}$$

where $(S_t)_{t\in\mathbb{R}_+}$ and $(S_t^*)_{t\in\mathbb{R}_+}$ are jump-diffusion price processes of the form

$$\frac{dS_t}{S_{t^-}} = r_t dt + \sigma_t dW_t + J_{t^-}(dZ_t - \lambda_t dt),$$

where $(W_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion, $(Z_t)_{t\in\mathbb{R}_+}$ is a point process of (stochastic) intensity λ_t, and $(S_t^*)_{t\in\mathbb{R}_+}$ can be taken as the solution of

$$\frac{dS_t^*}{S_{t^-}^*} = r_t dt + \sigma_t^* d\hat{W}_t + J_{t^-}^*(d\hat{N}_t - \lambda_t^* dt),$$

where $(\hat{W}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion and $(\hat{N}_t)_{t\in\mathbb{R}_+}$ is a Poisson process of (deterministic) intensity λ_t^*, mutually independent and independent of $(W_t)_{t\in\mathbb{R}_+}$ and of $(N_t)_{t\in\mathbb{R}_+}$, provided the three conditions

$$|\sigma_t| \leq |\sigma_t^*|, \qquad 0 \leq J_t \leq J_t^*, \qquad J_t\lambda_t \leq J_t^*\lambda_t^*, \qquad t \in [0, T], \tag{1.5}$$

are satisfied. The choice of the standard Poisson process $(\hat{N}_t)_{t\in\mathbb{R}_+}$ to drive the jump part of $(S_t^*)_{t\in\mathbb{R}_+}$ is made here to simplify the formulation of the hypotheses in (1.5). More general point processes can be actually considered, cf. Sections 4, 5 and 6.

Here, the coefficients $r_t,\ \sigma_t,\ J_t,\ \sigma_t^*,\ J_t^*$, are (random) $\mathcal{F}^M$-adapted processes and need not be diffusion coefficients. The difference between (1.4) and (1.1) is that in (1.4) the integrator processes $(\hat{W}_t)_{t\in\mathbb{R}_+}$ and $(\hat{N}_t)_{t\in\mathbb{R}_+}$ are independent of $(\sigma_t^*)_{t\in\mathbb{R}_+}$ and of $(J_t^*)_{t\in\mathbb{R}_+}$.

Denoting by $BS(\phi,x,t,r^*,\sigma^*,J^*)$ the conditional Black-Scholes price

$$BS(\phi,x,t,r,\sigma^*,J^*) = E[\phi(S_t^*)|\ W,\ Z,\ S_0^* = x],$$

(1.4) reads

$$E[\phi(S_t) \mid S_0 = x] \le E[BS(\phi,x,t,r^*,\sigma^*,J^*)]$$

between $E[\phi(S_t) \mid S_0 = x]$ and the averaged Black-Scholes price

$$E[BS(\phi,x,t,r^*,\sigma^*,J^*)].$$

In the diffusion case when $J^* = \lambda^* = 0$ and σ_t^* is deterministic, our result coincides with those of the above mentioned papers, and in particular with (1.1) or Theorem 6.2 of [4].

In the jump-diffusion case, still taking σ_t^*, J_t^*, λ_t^* deterministic, we get

$$E[\phi(S_T) \mid S_0 = x] \le BS(\phi,x,T,r^*,\sigma^*,J^*),$$

but our hypothesis differ from those of [2] where convex ordering of the jump caracteristics is required, see Theorem 2.3 therein, whereas here our conditions are directly formulated in terms of $J_t,\ \lambda_t,\ J_t^*$ and λ_t^*. In the general case where σ_t^*, J_t^*, λ_t^* are random, our results can not be compared since our process $(S_t^*)_{t\in\mathbb{R}_+}$ is no longer a diffusion process as in [2].

We proceed as follows. In Section 2 we recall the framework of [6] on forward-backward stochastic calculus. In Section 3 we prove our convex concentration inequalities for exponential martingales following the arguments of [6], in which sums are replaced by products. Applications to point processes and Poisson random measures in view of option pricing are given in Sections 4, 5, 6.

2. Forward-backward stochastic calculus

In this section we recall some definitions and results on forward-backward stochastic calculus, see [6] for details. Let $(\Omega,\mathcal{F},P)$ be a probability space equipped with an increasing filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ and a decreasing filtration $(\mathcal{F}_t^*)_{t\in\mathbb{R}_+}$. Consider $(X_t)_{t\in\mathbb{R}_+}$ an $\mathcal{F}_t$-forward martingale with $X_0 = 0$, and $(X_t^*)_{t\in\mathbb{R}_+}$ an $\mathcal{F}_t^*$-backward martingale with $X_\infty^* = 0$, such that $(X_t)_{t\in\mathbb{R}_+}$ has right-continuous paths with left limits and $(X_t^*)_{t\in\mathbb{R}_+}$ has left-continuous paths with right limits. Denote respectively by $(X_t^c)_{t\in\mathbb{R}_+}$ and $(X_t^{*c})_{t\in\mathbb{R}_+}$ the continuous parts of $(X_t)_{t\in\mathbb{R}_+}$ and of $(X_t^*)_{t\in\mathbb{R}_+}$, and by

$$\Delta X_t = X_t - X_{t^-}, \qquad \Delta^* X_t^* = X_t^* - X_{t^+}^*,$$

their forward and backward jumps. Denote by

$$\mu(dt,dx) = \sum_{s>0} 1_{\{\Delta X_s \neq 0\}} \delta_{(s,\Delta X_s)}(dt,dx),$$

and

$$\mu^*(dt,dx) = \sum_{s>0} 1_{\{\Delta^* X_s^* \neq 0\}} \delta_{(s,\Delta^* X_s^*)}(dt,dx),$$

the jump measures of $(X_t)_{t\in\mathbb{R}_+}$ and $(X^*_t)_{t\in\mathbb{R}_+}$, by $\nu(dt,dx)$ and $\nu^*(dt,dx)$ their respective $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ and $(\mathcal{F}^*_t)_{t\in\mathbb{R}_+}$-dual predictable projections, and by $([X,X])_{t\in\mathbb{R}_+}$, $([X^*,X^*])_{t\in\mathbb{R}_+}$, resp. $\langle X^c,X^c\rangle_t$, $\langle X^{*c},X^{*c}\rangle_t$ $t\in\mathbb{R}_+$, their optional, resp. predictable quadratic variations, which constitute the local characteristics of $(X_t)_{t\in\mathbb{R}_+}$, cf. [5] in the forward case.

We will use the following Itô type change of variable formula for forward-backward martingales which has been proved in [6], Theorem 8.1:

$$\begin{aligned} & f(X_t,X^*_t) - f(X_0,X^*_0) \\ &= \int_{0^+}^t \partial_1 f(X_{u^-},X^*_u)dX_u + \frac{1}{2}\int_0^t \partial_1^2 f(X_u,X^*_u)d\langle X^c,X^c\rangle_u \\ &\quad + \sum_{0<u\le t}\left(f(X_u,X^*_u) - f(X_{u^-},X^*_u) - \Delta X_u \partial_1 f(X_{u^-},X^*_u)\right) \\ &\quad - \int_0^{t^-} \partial_2 f(X_u,X^*_{u^+})d^*X^*_u - \frac{1}{2}\int_0^t \partial_2^2 f(X_u,X^*_u)d\langle X^{*c},X^{*c}\rangle_u \\ &\quad - \sum_{0\le u<t}\left(f(X_u,X^*_u) - f(X_u,X^*_{u^+}) - \Delta X^*_u \partial_2 f(X_u,X^*_{u^+})\right), \end{aligned}$$

for all $f\in C^2(\mathbb{R}^2,\mathbb{R})$, where d^* denotes the backward Itô differential.

Finally, recall the following classical comparison lemma.

Lemma 2.1. *Let m_1, m_2 be two finite non-negative measures on $\mathbb{R}$. The inequality*

$$\int_{-\infty}^{\infty} f(x)m_1(dx) \le \int_{-\infty}^{\infty} f(x)m_2(dx)$$

holds for all non-decreasing m_1, m_2-integrable function f on $\mathbb{R}$ if and only if

$$m_1([x,\infty)) \le m_2([x,\infty)),$$

for all $x\in\mathbb{R}$.

3. Convex comparison for exponential martingales

In the sequel we will assume further that the angle brackets of $(X_t)_{t\in\mathbb{R}_+}$ and $(X^*_t)_{t\in\mathbb{R}_+}$ have the form

$$d\langle X^c,X^c\rangle_t = |\sigma_t|^2dt, \quad \text{and} \quad d\langle X^{*c},X^{*c}\rangle_t = |\sigma^*_t|^2dt, \tag{3.1}$$

and jump measures of the form

$$\nu(dt,dx) = \nu_t(dx)dt \qquad \text{and} \qquad \nu^*(dt,dx) = \nu^*_t(dx)dt, \tag{3.2}$$

where $(\sigma_t)_{t\in\mathbb{R}_+}$, $(\nu_t)_{t\in\mathbb{R}_+}$ and $(\sigma^*_t)_{t\in\mathbb{R}_+}$, $(\nu^*_t)_{t\in\mathbb{R}_+}$, are respectively predictable with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ and to $(\mathcal{F}^*_t)_{t\in\mathbb{R}_+}$.

Let now $(\mathcal{F}^X_t)_{t\in\mathbb{R}_+}$, resp. $(\mathcal{F}^{X^*}_t)_{t\in\mathbb{R}_+}$, denote the forward, resp. backward, filtration generated by $(X_t)_{t\in\mathbb{R}_+}$, resp. by $(X^*_t)_{t\in\mathbb{R}_+}$. Recall that

$$(X_t)_{t\in\mathbb{R}_+},\ (X^*_t)_{t\in\mathbb{R}_+} \text{ are respectively } \mathcal{F}^*_t\text{-adapted and } \mathcal{F}_t\text{-adapted.} \tag{3.3}$$

Let $M = \mathcal{E}(X)$ and $M^* = \mathcal{E}^*(X^*)$ respectively denote the forward and backward stochastic exponentials of X and of X^*, i.e.

$$M_t = M_0 \exp\left(X_t - \frac{1}{2}[X,X]_t\right) \prod_{0\le s\le t} \left((1+\Delta X_{s^-})e^{-\Delta X_{s^-}+\frac{1}{2}|\Delta X_{s^-}|^2}\right),$$

in the forward case, cf. [7], and

$$M_t^* = M_\infty^* \exp\left(X_t^* - \frac{1}{2}\int_t^\infty d[X^*,X^*]_s\right) \prod_{t\le s<\infty}\left((1+\Delta^* X_{s^+}^*)e^{-\Delta^* X_{s^+}^*+\frac{1}{2}|\Delta^* X_{s^+}^*|^2}\right),$$

$t \in \mathbb{R}_+$, in the backward case. Equivalently, $(M_t)_{t\in\mathbb{R}_+}$ and $(M_t^*)_{t\in\mathbb{R}_+}$ are the respective solutions of

$$dM_t = M_{t^-} dX_t \qquad \text{and} \qquad d^*M_t^* = M_{t^+}^* d^* X_t^*,$$

with respective initial and final conditions M_0 and M_∞^*. Note also that $(M_t)_{t\ge 0}$ and $(M_t^*)_{t\ge 0}$ do not need to be independent. Let now

$$\bar\nu_t(dx) := x\nu_t(dx), \qquad \bar\nu_t^*(dx) := x\nu_t^*(dx), \qquad t\in\mathbb{R}_+,$$

and

$$\tilde\nu_t(dx) := |x|^2\nu_t(dx) + |\sigma_t|^2\delta_0(dx), \qquad \tilde\nu_t^*(dx) := |x|^2\nu_t^*(dx) + |\sigma_t^*|^2\delta_0(dx),$$

$t \in \mathbb{R}_+$.

Theorem 3.1. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex function. Assume that* (3.1)*,* (3.2)*,* (3.3) *hold, and either:*

i) $|\sigma_t| \le |\sigma_t^*|$, $dPdt$-a.e., ν_t and ν_t^* are supported by $\mathbb{R}_+$, $M_0, M_\infty^* \ge 0$, a.s., and

$$\bar\nu_t([x,\infty) \le \bar\nu_t^*([x,\infty)) < \infty, \qquad x\in\mathbb{R},\ t\in\mathbb{R}_+,$$

or:

ii) $\tilde\nu_t([x,\infty)) \le \tilde\nu_t^*([x,\infty)) < \infty$, $x \in \mathbb{R}$, $t \in \mathbb{R}_+$ and ϕ' is convex.
Then we have:

$$E[\phi(M_t M_t^*)] \le E[\phi(M_s M_s^*)], \qquad 0\le s\le t.$$

If in addition

iii) $E[M_t^* | \mathcal{F}_t^X] = 1$, $t \in \mathbb{R}_+$,

then

$$E[\phi(M_t)] \le E[\phi(M_s M_s^*)], \qquad 0 \le s \le t. \tag{3.4}$$

Proof. By an approximation argument we may assume that ϕ is $\mathcal{C}^2$ and convex. We apply Itô's formula for forward-backward martingales to $f(x_1, x_2) = \phi(x_1 x_2)$:

$$\begin{aligned}
f(M_t M_t^*) &= f(M_s M_s^*) \\
&\quad + \int_{s^+}^t M_u M_u^* \phi'(M_{u^-} M_u^*) dX_u + \frac{1}{2}\int_s^t (M_u M_u^*)^2 \phi''(M_u M_u^*) d\langle X^c, X^c\rangle_u \\
&\quad + \sum_{s<u\le t} (\phi((1+\Delta X_u) M_{u^-} M_u^*)) - \phi(M_{u^-} M_u^*) - M_u^* M_{u^-} \Delta X_u \phi'(M_{u^-} M_u^*)) \\
&\quad - \int_s^{t^-} M_u M_u^* \phi'(M_u M_{u^+}^*) d^* X_u^* - \frac{1}{2}\int_s^t (M_u M_u^*)^2 \phi''(M_u M_u^*) d\langle X^{*c}, X^{*c}\rangle_u \\
&\quad - \sum_{s\le u<t} (\phi(M_u M_{u^+}^* (1+\Delta^* X_u^*)) - \phi(M_u M_{u^+}^*) - M_u M_{u^+}^* \Delta^* X_u^* \phi'(M_u, M_{u^+}^*)),
\end{aligned}$$

$0 \le s \le t$, where d and d^* denote the forward and backward Itô differential.

i) Under hypothesis (*i*), taking expectations on both sides of this formula we get

$$\begin{aligned}
&E[\phi(M_t M_t^*)] = E[\phi(M_s M_s^*)] + \\
&\frac{1}{2} E\left[\int_s^t (M_u M_u^*)^2 \phi''(M_u M_u^*) d\langle X^c, X^c\rangle_u - \int_s^t (M_u M_u^*)^2 \phi''(M_u M_u^*) d\langle X^{*c}, X^{*c}\rangle_u\right] \\
&+E\left[\int_s^t \int_{-\infty}^\infty \big(\phi(M_u M_u^*(1+x)) - \phi(M_u M_u^*) - M_u^* M_u x \phi'(M_u M_u^*)\big)\nu_u(dx) du\right] \\
&-E\left[\int_s^t \int_{-\infty}^\infty \big(\phi(M_u M_u^*(1+x)) - \phi(M_u M_u^*) - M_u^* M_u x \phi'(M_u M_u^*)\big)\nu_u^*(dx) du\right] \\
&\quad = \quad E[\phi(M_s M_s^*)] + \frac{1}{2} E\left[\int_s^t (M_u M_u^*)^2 \phi''(M_u M_u^*) \big(|\sigma_u|^2 - |\sigma_u^*|^2\big) du\right] \\
&\qquad\quad + E\left[\int_s^t M_u M_u^* \int_{-\infty}^\infty \varphi(M_u M_u^* x, M_u M_u^*) \big(\bar\nu_u(dx) - \bar\nu_u^*(dx)\big) du\right],
\end{aligned}$$

where

$$\varphi(x, y) = \frac{\phi(x+y) - \phi(y) - x\phi'(y)}{x}, \qquad x \in \mathbb{R}_+, \quad y \in \mathbb{R}.$$

By the comparison Lemma 2.1, the conclusion follows from the hypothesis and the fact that since ϕ is convex, the function $x \mapsto \varphi(x, y)$ is nondecreasing in $x \in \mathbb{R}_+$ for all $y \in \mathbb{R}$, and $\phi''(x) \ge 0$ for all $x \in \mathbb{R}$.

ii) Under hypothesis (*ii*), using the identity

$$\phi(y(1+x)) = \phi(y) + xy\phi'(y) + |xy|^2 \int_0^1 (1-\tau)\phi''(y(1+\tau x)) d\tau, \qquad x, y \in \mathbb{R},$$

we have:

$$\begin{aligned} E[\phi(M_tM_t^*)] &= E[\phi(M_sM_s^*)] \\ &+\frac{1}{2}E\left[\int_s^t (M_uM_u^*)^2\phi''(M_uM_u^*)\big(|\sigma_u|^2-|\sigma_u^*|^2\big)du\right] \\ &+E\left[\int_s^t (M_uM_u^*)^2\int_{-\infty}^{\infty}|x|^2\int_0^1(1-\tau)\phi''(M_uM_u^*(1+\tau x))d\tau\nu_u(dx)du\right] \\ &-E\left[\int_s^t (M_uM_u^*)^2\int_{-\infty}^{\infty}|x|^2\int_0^1(1-\tau)\phi''(M_uM_u^*(1+\tau x))d\tau\nu_u^*(dx)du\right] \\ &= E[\phi(M_sM_s^*)] \\ &+E\left[\int_0^1(1-\tau)\int_s^t(M_uM_u^*)^2\int_{-\infty}^{\infty}\phi''(M_uM_u^*(1+\tau x))\big(\tilde\nu_u(dx)-\tilde\nu_u^*(dx)\big)dud\tau\right], \end{aligned}$$

and the conclusion follows from the hypothesis, the use of Lemma 2.1 and the use of the fact that ϕ'' is non-decreasing when ϕ' is convex.

iii) Under the additional hypothesis (*iii*), the proof of (3.4) follows from the argument leading to (1.3) in the introduction. □

We close this section with the following remarks, which will be useful in particular to take into account interest rate processes in financial applications.

Remark 3.2. i) The proofs and statements of Theorem 3.1 extend to semimartingales $(M_t)_{t\in\mathbb{R}_+}$, $(M_t^*)_{t\in\mathbb{R}_+}$ solutions of

$$\frac{dM_t}{M_{t^-}} = r_tdt + dX_t \quad \text{and} \quad \frac{d^*M_t^*}{M_{t^+}^*} = r_tdt + d^*X_t,$$

provided $(r_t)_{t\in\mathbb{R}_+}$ is both $\mathcal{F}_t$ and $\mathcal{F}_t^*$-adapted.
ii) If ϕ is non-decreasing, the same bounds hold also for semimartingales $(M_t)_{t\in\mathbb{R}_+}$, $(M_t^*)_{t\in\mathbb{R}_+}$ represented as

$$\frac{dM_t}{M_{t^-}} = r_tdt + dX_t \quad \text{and} \quad \frac{d^*M_t^*}{M_{t^+}^*} = r_t^*dt + d^*X_t,$$

provided $(r_t)_{t\in\mathbb{R}_+}$, $(r_t^*)_{t\in\mathbb{R}_+}$, are respectively $\mathcal{F}_t$ and $\mathcal{F}_t^*$-adapted with $r_t \le r_t^*$, $dPdt$-a.e.

Note also that all upper bounds presented in this paper can be stated as lower bounds provided the order of the inequalities on the process characteristics is reversed in the assumptions.

4. Bounded jumps

Here we assume that $\nu_t^*(dx)$ has the form

$$\nu_t^*(dx) = \lambda_t^*\delta_k(dx)dt, \tag{4.1}$$

where $(\lambda_t^*)_{t\in\mathbb{R}_+}$ is a positive $\mathcal{F}_t^*$-predictable process, i.e. $(X_t^*)_{t\in\mathbb{R}_+}$ has jumps of fixed size $k\in\mathbb{R}$. In this setting we have

$$M_t^* = 1 + \int_t^\infty \sigma_s^*M_s^*d^*W_s^* + k\int_t^\infty M_{s^+}(d^*Z_s^* - \lambda_s^*ds),$$

or equivalently

$$M_t^* = \exp\Big(\int_t^\infty \sigma_s^* d^* W_s^* - \frac{1}{2}\int_t^\infty |\sigma_s^*|^2 ds - k\int_t^\infty \lambda_s^* ds\Big) \prod_{t\le s<T} (1 + k\Delta^* Z_s^*),$$

$t \in \mathbb{R}_+$, where $(W_t^*)_{t\in\mathbb{R}_+}$ is a backward Brownian motion and $(Z_t^*)_{t\in\mathbb{R}_+}$ is a backward point process with intensity $(\lambda_t^*)_{t\in\mathbb{R}_+}$.

Corollary 4.1. *Assume that* (3.1), (3.2), (3.3) *and* (4.1) *hold, and that one of the following conditions is satisfied:*
i) $0 \le \Delta X_t \le k$, *dPdt-a.e. and*

$$|\sigma_t| \le |\sigma_t^*|, \qquad \int_0^k x\nu_t(dx) \le k\lambda_t^*, \qquad dPdt - a.e.$$

ii) $\Delta X_t \le k$, *dPdt-a.e. and*

$$|\sigma_t| \le |\sigma_t^*|, \qquad \int_{-\infty}^k |x|^2\nu_t(dx) \le k^2\lambda_t^*, \qquad dPdt - a.e.$$

iii) $\Delta X_t \le 0 \le k$, *dPdt-a.e. and*

$$|\sigma_t|^2 + \int_{-\infty}^0 |x|^2\nu_t(dx) \le |\sigma_t^*|^2 + k^2\lambda_t^*, \qquad dPdt - a.e.$$

iv) $0 \le \Delta X_t \le k$, *dPdt-a.e.* $\int_0^k |x|^2\nu_t(dx) \le k^2\lambda_t^*$ *and*

$$|\sigma_t|^2 + \int_0^k |x|^2\nu_t(dx) \le |\sigma_t^*|^2 + k^2\lambda_t^*, \qquad dPdt - a.e.$$

v) $\Delta X_t \le k \le 0$, *dPdt-a.e.* $|\sigma_t| \le |\sigma_t^*|$, *dPdt-a.e., and*

$$|\sigma_t|^2 + \int_{-\infty}^k |x|^2\nu_t(dx) \le |\sigma_t^*|^2 + k^2\lambda_t^*, \qquad dPdt - a.e.$$

Then we have:

$$E[\phi(M_t)] \le E[\phi(M_s M_s^*)], \qquad 0 \le s \le t, \tag{4.2}$$

for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$, *with moreover* ϕ' *convex in cases* $ii)$–$v)$.

Proof. The condition $0 \le \Delta X_t \le k$, resp. $\Delta X_t \le k$, is equivalent to $\nu_t([0,k]^c) = 0$, resp. $\nu_t((k,\infty)) = 0$. Using the expressions

$$\begin{aligned} \bar{\nu}_t^*([x,+\infty[) &= k\lambda_t^* \mathbf{1}_{]-\infty,k]}(x), \\ \tilde{\nu}_t^*([x,+\infty[) &= k^2\lambda_t^* \mathbf{1}_{]-\infty,k]}(x) + |\sigma_t^*|^2\delta_0([x,+\infty[), \end{aligned}$$

and the comparison Lemma 2.1 we show that the hypothesis of Theorem 3.1-(i), resp. (ii), are satisfied in case (i), resp. in cases (ii)–(v). □

Note that in Corollary 4.1 and in the following corollaries, conditions (iii)-(iv) make a combined use of the continuous and jump characteristics of the processes.

Assume now that $\mathcal{F}^X$ is generated by a standard Brownian motion $(W_t)_{t\in\mathbb{R}_+}$ and a point process $(J_t)_{t\in\mathbb{R}_+}$ with intensity $(\lambda_t)_{t\in\mathbb{R}_+}$. Let $(S_t)_{t\in\mathbb{R}_+}$ be a price process defined by

$$\frac{dS_t}{S_{t^-}} = r_t dt + \sigma_t dW_t + J_{t^-}(dZ_t - \lambda_t dt), \tag{4.3}$$

where $(r_t)_{t\in\mathbb{R}_+}$, $(\sigma_t)_{t\in\mathbb{R}_+}$, $(J_t)_{t\in\mathbb{R}_+}$ are square-integrable $\mathcal{F}_t^X$-adapted processes. Consider $(\hat{W})_{t\in\mathbb{R}_+}$ a (forward) standard Brownian motion and $(\hat{N}_t)_{t\in\mathbb{R}_+}$ a (forward, right-continuous) Poisson process of (deterministic) intensity $(\hat{\lambda}_t)_{t\in\mathbb{R}_+}$, both independent of $\mathcal{F}^X$, and let $(S_t^*)_{t\in\mathbb{R}_+}$ be defined by

$$\frac{dS_t^*}{S_{t^-}^*} = r_t dt + \sigma_t^* d\hat{W}_t + k(d\hat{N}_t - \hat{\lambda}_t dt), \tag{4.4}$$

where $(\sigma_t^*)_{t\in\mathbb{R}_+}$ is a square-integrable $\mathcal{F}_t^X$-adapted process.

Corollary 4.2. *Assume that one of the following conditions is satisfied:*
i) $0 \le \Delta X_s \le k$, *$dPds$-a.e. and*

$$|\sigma_s| \le |\sigma_s^*|, \qquad \int_0^k x\nu_s(dx) \le k\hat{\lambda}_s, \qquad dPds - a.e.$$

ii) $\Delta X_s \le k$, *$dPds$-a.e. and*

$$|\sigma_s| \le |\sigma_s^*|, \qquad \int_{-\infty}^k |x|^2\nu_s(dx) \le k^2\hat{\lambda}_s, \qquad dPds - a.e.$$

iii) $\Delta X_s \le 0 \le k$, *$dPds$-a.e. and*

$$|\sigma_s|^2 + \int_{-\infty}^0 |x|^2\nu_s(dx) \le |\sigma_s^*|^2 + k^2\hat{\lambda}_s, \qquad dPds - a.e.$$

iv) $0 \le \Delta X_s \le k$, *$dPds$-a.e.*, $\int_0^k |x|^2\nu_s(dx) \le k^2\hat{\lambda}_s$, *$dPds$-a.e. and*

$$|\sigma_s|^2 + \int_0^k |x|^2\nu_s(dx) \le |\sigma_s^*|^2 + k^2\hat{\lambda}_s, \qquad dPds - a.e.$$

v) $\Delta X_s \le k \le 0$, *$dPds$-a.e.*, $|\sigma_s| \le |\sigma_s^*|$, *$dPds$-a.e. and*

$$|\sigma_s|^2 + \int_{-\infty}^k |x|^2\nu_s(dx) \le |\sigma_s^*|^2 + k^2\hat{\lambda}_s, \qquad dPds - a.e.$$

Then we have

$$E[\phi(S_t) \mid S_0 = x] \le E[\phi(S_t^*) \mid S_0^* = x], \qquad x > 0, \quad t \in \mathbb{R}_+, \tag{4.5}$$

with moreover ϕ' convex in cases (ii)–(v).

Proof. We apply Corollary 4.1 to the processes $(1_{[0,t]}(s)\sigma_s)_{s\in\mathbb{R}_+}$, $(1_{[0,t]}(s)\sigma_s^*)_{s\in\mathbb{R}_+}$, $(1_{[0,t]}(s)\nu_s)_{s\in\mathbb{R}_+}$, $(1_{[0,t]}(s)\lambda_s)_{s\in\mathbb{R}_+}$, $(\lambda_s^*)_{s\in\mathbb{R}_+} := (1_{[0,t]}(s)\hat{\lambda}_s)_{s\in\mathbb{R}_+}$, with $M_t^* = 1$, and use the identity in law

$$M_0^* = S_t^*,$$

which holds because the forward and backward stochastic integral coincide when the integrator process is independent of the integrand. □

5. Point process diffusions

Consider now $(W_t)_{t\in\mathbb{R}_+}$, $(Z_t)_{t\in\mathbb{R}_+}$ a standard Brownian motion and a point process generating the filtration

$$\mathcal{F}_t^X = \sigma(W_s, Z_s \ : \ 0 \le s \le t), \qquad t \in \mathbb{R}_+.$$

We will assume that $(W_t)_{t\in\mathbb{R}_+}$ is also an $\mathcal{F}_t^X$-Brownian motion and that $(Z_t)_{t\in\mathbb{R}_+}$ has compensator $(\lambda_t)_{t\in\mathbb{R}_+}$ with respect to $(\mathcal{F}_t^X)_{t\in\mathbb{R}_+}$, which does not in general require the independence of $(W_t)_{t\in\mathbb{R}_+}$ from $(Z_t)_{t\in\mathbb{R}_+}$.

Consider also $(W_t^*)_{t\in\mathbb{R}_+}$ a backward standard Brownian motion, $(Z_t^*)_{t\in\mathbb{R}_+}$ a backward point process with intensity $(\lambda_t^*)_{t\in\mathbb{R}_+}$, all independent of $(\mathcal{F}_t^X)_{t\in\mathbb{R}_+}$, and let $(X_t)_{t\in\mathbb{R}_+}$ and $(X_t^*)_{t\in\mathbb{R}_+}$ be defined as

$$X_t = \int_0^t \sigma_s dW_s + \int_0^t J_{s^-}(dZ_s - \lambda_s ds), \quad t \in \mathbb{R}_+,$$

and

$$X_t^* = \int_t^\infty \sigma_s^* d^* W_s^* + \int_t^\infty J_{s^+}^*(d^* Z_s^* - \lambda_s^* ds), \quad t \in \mathbb{R}_+,$$

where σ_t is a square-integrable $\mathcal{F}_t^X$-predictable process and $(J_t)_{t\in\mathbb{R}_+}$ is an $\mathcal{F}_t^X$-predictable process which is either square-integrable or positive and integrable, and the processes $(\sigma_t^*)_{t\in\mathbb{R}_+}$, $(J_t^*)_{t\in\mathbb{R}_+}$, are taken (forward) $\mathcal{F}_t^X$-predictable under the same integrability conditions. The characteristic measures of X and X^* are given by

$$\nu_t(dx) = \lambda_t \delta_{J_t}(dx)dt \quad \text{and} \quad \nu_t^*(dx) = \lambda_t^* \delta_{J_t^*}(dx)dt$$

and we have

$$M_t = M_0 + \int_0^t \sigma_s M_s dW_s + \int_0^t J_{s^-} M_{s^-}(dZ_s - \lambda_s ds), \quad t \in \mathbb{R}_+,$$

and

$$M_t^* = 1 + \int_t^\infty \sigma_s^* M_s^* d^* W_s^* + \int_t^\infty J_{s^+}^* M_{s^+}^*(d^* Z_s^* - \lambda_s^* ds), \quad t \in \mathbb{R}_+, \tag{5.1}$$

i.e.

$$M_t = M_0 \exp\Big(\int_0^t \sigma_s dW_s - \frac{1}{2}\int_0^t \sigma_s^2 ds - \int_0^t \lambda_s ds\Big) \prod_{0<s\le t}(1 + J_{s^-}\Delta Z_s),\ t \in \mathbb{R}_+,$$

and

$$M_t^* = \exp\Big(\int_t^\infty \sigma_s^* d^* W_s^* - \frac{1}{2}\int_t^\infty |\sigma_s^*|^2 ds - \int_t^\infty J_s^* \lambda_s^* ds\Big) \prod_{t\le s<\infty}(1 + J_{s^+}^* \Delta^* Z_s^*),$$

$t \in \mathbb{R}_+$. Applying Theorem 3.1 with

$$\mathcal{F}_t = \mathcal{F}_t^X \vee \sigma(W_s^*, Z_s^* \ : \ s \in \mathbb{R}_+), \qquad t \in \mathbb{R}_+,$$

and

$$\mathcal{F}_t^* = \mathcal{F}_\infty^X \vee \sigma(W_s^*, Z_s^* \ : \ s \ge t), \qquad t \in \mathbb{R}_+,$$

and noting that $E[M_t^* | \mathcal{F}_t^X] = 1$ (as follows from the independence of $(W_t^*)_{t\in\mathbb{R}_+}$ and of $(Z_t^*)_{t\in\mathbb{R}_+}$ with $\mathcal{F}^X$), $t \in \mathbb{R}_+$, we derive the following corollary:

Corollary 5.1. *Assume that* (3.1), (3.2), (3.3) *hold. Then we have:*

$$E[\phi(M_t)] \le E[\phi(M_s M_s^*)], \qquad 0 \le s \le t,$$

for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$*, provided one of the following conditions is satisfied:*
i) $0 \le J_t \le J_t^*$, $\lambda_t dPdt$*-a.e. and*

$$|\sigma_t| \le |\sigma_t^*|, \qquad \lambda_t J_t \le \lambda_t^* J_t^*, \qquad dPdt - a.e.,$$

ii) $J_t \le J_t^*$, $\lambda_t dPdt$*-a.e. and*

$$|\sigma_t| \le |\sigma_t^*|, \qquad \lambda_t |J_t|^2 \le \lambda_t^* |J_t^*|^2, \qquad dPdt - a.e.,$$

iii) $J_t \le 0 \le J_t^*$, $\lambda_t dPdt$*-a.e., and*

$$|\sigma_t|^2 + \lambda_t |J_t|^2 \le |\sigma_t^*|^2 + \lambda_t^* |J_t^*|^2, \qquad dPdt - a.e.,$$

iv) $0 \le J_t \le J_t^*$, $\lambda_t dPdt$*-a.e.,*

$$\lambda_t |J_t|^2 \le \lambda_t^* |J_t^*|^2, \quad \textit{and} \quad |\sigma_t|^2 + \lambda_t |J_t|^2 \le |\sigma_t^*|^2 + \lambda_t^* |J_t^*|^2, \qquad dPdt - a.e.,$$

v) $J_t \le J_t^* \le 0$, $\lambda_t dPdt$*-a.e.,*

$$|\sigma_t| \le |\sigma_t^*| \quad \textit{and} \quad |\sigma_t|^2 + \lambda_t |J_t|^2 \le |\sigma_t^*|^2 + \lambda_t^* |J_t^*|^2, \qquad dPdt - a.e.$$

with moreover ϕ' *convex in cases ii)–v).*

Proof. Similarly to the proof of Corollary 4.1, we use the comparison Lemma 2.1 and show that the hypothesis of Theorem 3.1-(i), resp. (ii), are satisfied in case (i), resp. in cases (ii)–(v), with

$$\begin{aligned}
\bar{\nu}_t([x,+\infty[) &= \lambda_t J_t \mathbf{1}_{]-\infty,J_t]}(x),\\
\bar{\nu}_t^*([x,+\infty[) &= \lambda_t^* J_t^* \mathbf{1}_{]-\infty,J_t^*]}(x),\\
\tilde{\nu}_t([x,+\infty[) &= \lambda_t |J_t|^2 \mathbf{1}_{]-\infty,J_t]}(x) + |\sigma_t|^2 \delta_0([x,+\infty[),\\
\tilde{\nu}_t^*([x,+\infty[) &= \lambda_t^* |J_t^*|^2 \mathbf{1}_{]-\infty,J_t^*]}(x) + |\sigma_t^*|^2 \delta_0([x,+\infty[).
\end{aligned}$$

□

In the following corollary, $(\hat{\lambda}_t)_{t\in\mathbb{R}_+}$ is a positive deterministic function and $(S_t)_{t\in\mathbb{R}_+}$, $(S_t^*)_{t\in\mathbb{R}_+}$ are respectively defined as in (4.3) and (4.4).

Corollary 5.2. *Assume that one of the following conditions is satisfied:*
i) $0 \le J_s \le J_s^*$, $\lambda_s dPds$*-a.e., and*

$$|\sigma_s| \le |\sigma_s^*|, \qquad \lambda_s J_s \le \hat{\lambda}_s J_s^*, \qquad dPds - a.e.$$

ii) $J_s \le J_s^*$, $\lambda_s dPds$*-a.e., and*

$$|\sigma_s| \le |\sigma_s^*|, \qquad \lambda_s |J_s|^2 \le \hat{\lambda}_s |J_s^*|^2, \qquad dPds - a.e.$$

iii) $J_s \le 0 \le J_s^*$, $\lambda_s dPds$*-a.e. and*

$$|\sigma_s|^2 + \lambda_s |J_s|^2 \le |\sigma_s^*|^2 + \hat{\lambda}_s |J_s^*|^2, \qquad dPds - a.e.$$

iv) $0 \le J_s \le J_s^*$, $\lambda_s dPds$*-a.e.* $\lambda_s |J_s|^2 \le \hat{\lambda}_s |J_s^*|^2$, $dPds$*-a.e. and*

$$|\sigma_s|^2 + \lambda_s |J_s|^2 \le |\sigma_s^*|^2 + \hat{\lambda}_s |J_s^*|^2, \qquad dPds - a.e.$$

v) $J_s \leq J_s^* \leq 0$, $\lambda_s dPds$*-a.e.* $|\sigma_s| \leq |\sigma_s^*|$, $dPds$*-a.e., and*

$$|\sigma_s|^2 + \lambda_s |J_s|^2 \leq |\sigma_s^*|^2 + \hat{\lambda}_s |J_s^*|^2, \qquad dPds - a.e.$$

Then we have

$$E[\phi(S_t) \mid S_0 = x] \leq E[\phi(S_t^*) \mid S_0^* = x], \qquad x > 0, \quad t \in \mathbb{R}_+, \tag{5.2}$$

for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$*, with moreover* ϕ' *convex in cases ii)–v).*

Proof. Similarly to the proof of Corollary 4.2, we apply Corollary 5.1 to the processes $(1_{[0,t]}(s)\sigma_s)_{s\in\mathbb{R}_+}$, $(1_{[0,t]}(s)\sigma_s^*)_{s\in\mathbb{R}_+}$, $(1_{[0,t]}(s)J_s)_{s\in\mathbb{R}_+}$, $(1_{[0,t]}(s)J_s^*)_{s\in\mathbb{R}_+}$, $(1_{[0,t]}(s)\lambda_s)_{s\in\mathbb{R}_+}$, with $(\lambda_s^*)_{s\in\mathbb{R}_+} := (1_{[0,t]}(s)\hat{\lambda}_s)_{s\in\mathbb{R}_+}$ and $M_t^* = 1$. □

6. Poisson random measures

We now investigate the consequences of Theorem 3.1 in the setting of Poisson random measures. Let γ be a diffuse Radon measure on $\mathbb{R}^d \setminus \{0\}$ with

$$\int_{\mathbb{R}^d\setminus\{0\}} (|x|^2 \wedge 1)\gamma(dx) < \infty,$$

and consider a random measure $\omega(dt, dx)$ of the form

$$\omega(dt, dx) = \sum_{i\in\mathbb{N}} \delta_{(t_i,x_i)}(dt, dx)$$

identified to its (locally finite) support $\{(t_i, x_i)\}_{i\in\mathbb{N}}$. We assume that $\omega(dt, dx)$ is Poisson distributed with intensity $dt\gamma(dx)$ on $\mathbb{R}_+ \times (\mathbb{R}^d \setminus \{0\})$, and consider a standard Brownian motion $(W_t)_{t\in\mathbb{R}_+}$, independent of $\omega(dt, dx)$, under a probability P on Ω. Let

$$\mathcal{F}_t^X = \sigma\big(W_s,\ \omega([0, s] \times A) \ :\ 0 \leq s \leq t,\ A \in \mathcal{B}_b(\mathbb{R}^d \setminus \{0\})\big), \qquad t \in \mathbb{R}_+,$$

where $\mathcal{B}_b(\mathbb{R}^d \setminus \{0\}) = \{A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}) \ :\ \gamma(A) < \infty\}$. Here, $(M_t)_{t\in\mathbb{R}_+}$ is the solution of the forward stochastic differential equation

$$dM_t = \sigma_t M_t dW_t + \int_{\mathbb{R}^d\setminus\{0\}} J_{t^-,x} M_{t^-} (\omega(dt, dx) - \gamma(dx)), \tag{6.1}$$

where σ_t is a square-integrable $\mathcal{F}_t^X$-predictable process and $(J_{t,x})_{(t,x)\in\mathbb{R}_+\times(\mathbb{R}^d\setminus\{0\})}$ is an $\mathcal{F}_t^X$-predictable process satisfying the hypotheses of Corollary 6.1 below, and $(M_t^*)_{t\in\mathbb{R}_+}$ is defined as in (5.1). On the other hand, $(M_t^*)_{t\in\mathbb{R}_+}$ solves the backward stochastic differential equation

$$d^* M_t^* = \sigma_t^* M_t^* d^* W_t^* + J_{t^+}^* M_{t^+}^* (d^* Z_t^* - \lambda_t^* dt),$$

where $(W_t^*)_{t\in\mathbb{R}_+}$ is a backward standard Brownian motion and $(Z_t^*)_{t\in\mathbb{R}_+}$ a backward point process with intensity $(\lambda_t^*)_{t\in\mathbb{R}_+}$, independent of $(\mathcal{F}_t^X)_{t\in\mathbb{R}_+}$.

Corollary 6.1. *Assume that* (3.1), (3.2), (3.3) *hold, and that one of the following conditions is satisfied:*

i) $0 \leq J_{t,x} \leq J_t^*$, $dP\gamma(dx)dt$*-a.e.,*

$$|\sigma_t| \leq |\sigma_t^*|, \quad \textit{and} \quad \int_{\mathbb{R}^d\setminus\{0\}} J_{t,y}\gamma(dy) \leq \lambda_t^* J_t^*, \qquad dPdt - a.e.,$$

ii) $J_{t,x} \leq J_t^*$, $dP\gamma(dx)dt$*-a.e.,*

$$|\sigma_t| \leq |\sigma_t^*|, \quad \textit{and} \quad \int_{\mathbb{R}^d\setminus\{0\}} |J_{t,y}|^2 \gamma(dy) \leq \lambda_t^* |J_t^*|^2, \qquad dPdt - a.e.,$$

iii) $J_{t,x} \leq 0 \leq J_t^*$, $dP\gamma(dx)dt$*-a.e.,*

$$|\sigma_t|^2 + \int_{\mathbb{R}^d\setminus\{0\}} |J_{t,x}|^2 \gamma(dx) \leq |\sigma_t^*|^2 + \lambda_t^* |J_t^*|^2, \qquad dPdt - a.e.,$$

iv) $J_{t,x} \leq 0 \leq J_t^*$, $dP\gamma(dx)dt$*-a.e.,*

$$\int_{\mathbb{R}^d\setminus\{0\}} |J_{t,x}|^2 \gamma(dx) \leq \lambda_t^* |J_t^*|^2, \quad \textit{and} \quad |\sigma_t|^2 + \int_{\mathbb{R}^d\setminus\{0\}} |J_{t,x}|^2 \gamma(dx) \leq |\sigma_t^*|^2 + \lambda_t^* |J_t^*|^2,$$

$dPdt$*-a.e., v)* $J_{t,x} \leq 0 \leq J_t^*$, $dP\gamma(dx)dt$*-a.e.,*

$$|\sigma_t| \leq |\sigma_t^*|, \quad \textit{and} \quad |\sigma_t|^2 + \int_{\mathbb{R}^d\setminus\{0\}} |J_{t,x}|^2 \gamma(dx) \leq |\sigma_t^*|^2 + \lambda_t^* |J_t^*|^2, \qquad dPdt - a.e.,$$

with

$$(J_{t,x})_{(t,x)\in\mathbb{R}_+\times(\mathbb{R}^d\setminus\{0\})} \in L^1(\Omega \times \mathbb{R}_+ \times (\mathbb{R}^d \setminus \{0\}), dP \times dt \times d\gamma)$$

in case (i) *and with*

$$(J_{t,x})_{(t,x)\in\mathbb{R}_+\times(\mathbb{R}^d\setminus\{0\})} \in L^2(\Omega \times \mathbb{R}_+ \times (\mathbb{R}^d \setminus \{0\}), dP \times dt \times d\gamma)$$

in cases $(ii) - (v)$*.*

Then we have:

$$E[\phi(M_t)] \leq E[\phi(M_s M_s^*)], \qquad 0 \leq s \leq t, \tag{6.2}$$

for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$*, with moreover* ϕ' *convex in cases* (ii)*–*(v)*.*

Proof. We directly apply Theorem 3.1 instead of Corollary 5.1, with $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ and $(\mathcal{F}_t^*)_{t\in\mathbb{R}_+}$ defined again as

$$\mathcal{F}_t = \mathcal{F}_t^X \vee \sigma(W_s^*, Z_s^* \ : \ s \in \mathbb{R}_+), \qquad t \in \mathbb{R}_+,$$

and

$$\mathcal{F}_t^* = \mathcal{F}_\infty^X \vee \sigma(W_s^*, Z_s^* \ : \ s \geq t), \qquad t \in \mathbb{R}_+.$$

Here, $\nu_t(dx)$ denotes the image measure of $\gamma(dx)$ by the mapping $x \mapsto J_{t,x}$, $t \geq 0$, $\mu(dt, dx)$ is the image measure of $\omega(dt, dx)$ by $(s, y) \mapsto (s, J_{s,y})$, i.e.

$$\mu(dt, dx) = \sum_{\omega(\{(s,y)\})=1} \delta_{(s, J_{s,y})}(dt, dx),$$

and $\nu^*(t, x, dy) = \lambda^*(t, x)\delta_{J^*(t,x)}(dy)$.

Note that our hypotheses imply $\nu_t((J_t^*, \infty)) = 0$. For $p = 1, 2$, using the comparison Lemma 2.1, the hypotheses of Theorem 3.1 are satisfied since

$$\int_{\mathbb{R}^d\setminus\{0\}} J_{t,y}^p \mathbf{1}_{\{x \leq J_{t,y}\}} \gamma(dy) \leq \lambda_t^* \int_x^\infty y^p \delta_{J_t^*}(dy), \qquad x \geq 0, \quad t \in \mathbb{R}_+,$$

which follows from

$$J_{t,x} \leq J_t^*, \qquad dP\gamma(dx)dt - a.e.,$$

and

$$\int_{\mathbb{R}^d\setminus\{0\}} J^p_{t,y}\gamma(dy) \le \lambda^*_t (J^*_t)^p, \qquad dPdt - a.e.$$

For $p = 1$, resp. $p = 2$, this proves (6.2) under (i), resp. (ii). Finally, we note that conditions (iii)–(v) imply

$$\int_{\mathbb{R}^d\setminus\{0\}} |J_{t,y}|^2 \mathbf{1}_{\{x\le J_{t,y}\}}\gamma(dy) + |\sigma_t|^2\delta_0([x,+\infty[)$$
$$\le \lambda^*_t \int_x^\infty y^p \delta_{J^*_t}(dy) + |\sigma^*_t|^2\delta_0([x,+\infty[),$$

$x \in \mathbb{R}$, which allows us to conclude again via Theorem 3.1 and the comparison Lemma 2.1. Note that here also we have $E[M^*_t|\mathcal{F}^X_t] = 1$, $t \in \mathbb{R}_+$. □

Finally, as in Corollaries 4.2 and 5.2 we get, defining respectively $(S_t)_{t\in\mathbb{R}_+}$ and $(S^*_t)_{t\in\mathbb{R}_+}$ as in (4.3) and (4.4):

Corollary 6.2. *Assume that one of the following conditions is satisfied:*
i) $0 \le J_{s,x} \le J^*_s$, *$dPds$-a.e.,*

$$|\sigma_s| \le |\sigma^*_s|, \quad \textit{and} \quad \int_{\mathbb{R}^d\setminus\{0\}} J_{s,y}\gamma(dy) \le \hat{\lambda}_s J^*_s, \qquad dPds - a.e.,$$

ii) $J_{s,x} \le J^*_s$, *$dP\gamma(dx)ds$-a.e.,*

$$|\sigma_s| \le |\sigma^*_s|, \quad \textit{and} \quad \int_{\mathbb{R}^d\setminus\{0\}} |J_{s,y}|^2\gamma(dy) \le \hat{\lambda}_s |J^*_s|^2, \qquad dPds - a.e.,$$

iii) $J_{s,x} \le 0 \le J^*_s$, *$dP\gamma(dx)ds$-a.e., and*

$$|\sigma_s|^2 + \int_{\mathbb{R}^d\setminus\{0\}} |J_{s,x}|^2\gamma(dx) \le |\sigma^*_s|^2 + \lambda^*_s|J^*_s|^2, \qquad dPds - a.e.,$$

iv) $J_{s,x} \le 0 \le J^*_s$, *$dP\gamma(dx)ds$-a.e.,*

$$\int_{\mathbb{R}^d\setminus\{0\}} |J_{s,x}|^2\gamma(dx) \le \lambda^*_s|J^*_s|^2 \quad \textit{and} \quad |\sigma_s|^2 + \int_{\mathbb{R}^d\setminus\{0\}} |J_{s,x}|^2\gamma(dx) \le |\sigma^*_s|^2 + \lambda^*_s|J^*_s|^2,$$

$dPds$-a.e., v) $J_{s,x} \le 0 \le J^*_s$, *$dP\gamma(dx)ds$-a.e.,*

$$|\sigma_s| \le |\sigma^*_s| \quad \textit{and} \quad |\sigma_s|^2 + \int_{\mathbb{R}^d\setminus\{0\}} |J_{s,x}|^2\gamma(dx) \le |\sigma^*_s|^2 + \lambda^*_s|J^*_s|^2, \qquad dPds - a.e.,$$

with

$$(J_{s,x})_{(s,x)\in[0,t]\times(\mathbb{R}^d\setminus\{0\})} \in L^1(\Omega \times [0,t] \times (\mathbb{R}^d \setminus \{0\}), dP \times ds \times d\gamma)$$

in case (i) *and with*

$$(J_{s,x})_{(s,x)\in[0,t]\times(\mathbb{R}^d\setminus\{0\})} \in L^2(\Omega \times [0,t] \times (\mathbb{R}^d \setminus \{0\}), dP \times ds \times d\gamma)$$

in cases (ii)*-*(v)*.*

Then we have

$$E[\phi(S_t) \mid S_0 = x] \le E[\phi(S^*_t) \mid S^*_0 = x], \qquad x > 0, \quad t \in \mathbb{R}_+, \tag{6.3}$$

with moreover ϕ' *convex in cases ii)–v).*

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