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## SUPERPOSED ORNSTEIN-UHLENBECK PROCESSES

JULIUS N. ESUNGE\* AND JEAN AUGUSTE MUHAU

**ABSTRACT.** In this work we present and define the superposed Ornstein-Uhlenbeck processes by following the Barndorff-Nielsen approach. The Black-Scholes model is the most referenced model in financial mathematics, but due to its limitations, particularly its constant volatility, the Black-Scholes model fails to capture the nature of the data in finance. In order to overcome this limitation, we present the Barndorff-Nielsen and Shephard model with, as a volatility model, the superposed Ornstein-Uhlenbeck processes which is sum of independent Ornstein-Uhlenbeck processes driven by Lévy process. In order to show that a financial model with constant volatility is less reliable in finance than one with stochastic process as a volatility, we will simulate the trajectories of the Black-Scholes model and that of the Barndorff-Nielsen and Shephard model and show that the Barndorff-Nielsen and Shephard model provides a better fit than the Black-Scholes model.

### 1. Introduction

The Black-Scholes, Merton GBM Model is the most important model of reference in Financial mathematics, in particular for modeling of the stock price of an option. However, this famous model is considered by many mathematicians as a “big failure” because the volatility is considered constant, the model is Gaussian, the rentability does not depend on the instantaneous time but on a certain period and the fact that the path does not have jumps.

Guillaume Coqueret, a mathematician, suggested that since the Black-Scholes, Merton model is based on a Brownian Motion (the price expression being a GBM), the trajectory will be continuous without jumps and the price will be Gaussian. This is the reason why the Lévy process is used now in financial mathematics since it is not stationary normally distributed and has a càdlàg<sup>1</sup> trajectory. Nevertheless, using this process does not suppress all the failures of the Black-Scholes, Merton Model [2].

Stochastic volatility models play an important role in finance for the evaluation of a derivative asset as an option. This kind of model is an approach for the resolution of the failures of the Black-Scholes model which uses constant volatility [16]. To account for the limitations of the Black-Scholes, Merton model, mathematicians use the volatility process to replace the constant volatility. Barndorff-Nielsen

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<sup>1</sup>Right continuous with left limits coming from French Continue à droite et limité à gauche.

and Niel Shephard proposed in a paper “Realized power variation and stochastic volatility models”, a volatility model using high-frequency information. This theory is derived for a semimartingale with continuous local martingale term. They proposed two conditions, one on the mean process of the semimartingale and the other one on the volatility process for which the Ornstein-Uhlenbeck (OU) process verify these two conditions. [16]

Barndorff-Nielsen and Shephard proposed a model which corrects the Black-Scholes model with a volatility process which is an Ornstein-Uhlenbeck driven by Lévy process [7]. In this same way, they proposed a volatility model as a superposition of Ornstein-Uhlenbeck processes, which is a sum of independent Ornstein-Uhlenbeck driven by Lévy processes [15].

This project has three main sections. In the first section, we will study the Lévy process and Lévy integral which are the principal ingredients for the construction of Ornstein-Uhlenbeck process driven by Lévy process. Some properties on OU driven by Lévy process such as self-decomposability and stationarity are provided. The second section will be focused on the superposition of Ornstein-Uhlenbeck processes (supOU). The construction and definition follow Barndorff-Nielsen and will be provided where the volatility model defined by Barndorff-Nielsen and Shephard will be given. In the last section, a simulation of a Financial market will be given where we will compare the Barndorff-Nielsen and Shephard model (BN-S) to the Black-Scholes model (BS) by fitting data describing an asset price.

## 2. Ornstein-Uhlenbeck Process Driven by Lévy Process

**Definition 2.1** (Lévy Process [18]). Let  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$  be a filtered probability space, where  $\mathcal{F} = \mathcal{F}_T$  and the filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ . A càdlàg, adapted, real value stochastic process  $L := (L_t)_{t \in [0, t]}$  with  $L_0 = 0$  a.s is called Lévy process if the following holds:

- (i)  $L$  has independent increments;
- (ii)  $L$  has stationary increments;
- (iii)  $L$  is Locally continuous i.e.  $\lim_{s \rightarrow t} \mathbb{P}(|L_t - L_s| > \epsilon) = 0$ , for every  $t \in [0, T]$  and  $\epsilon > 0$ .

*Remark 2.2.* The Brownian Motion, Poisson Process (Compound Poisson process) and deterministic process (linear drift) are all Lévy processes.

**Definition 2.3.** We call Lévy jump-diffusion process a sum of a linear drift process, a Brownian motion, and a compound Poisson process [18].

*Remark 2.4.* The Lévy jump-diffusion process is martingale if the deterministic process is zero since the compound Poisson processes and Brownian motion are martingale.

**Proposition 2.5.** *The following Lévy process is a Lévy jump-diffusion process*

$$L_t = bt + \sigma B_t + \left( \sum_{k=1}^{N_t} J_k - t\lambda k \right),$$

where  $b \in \mathbb{R}$ ,  $\sigma \in \mathbb{R}_+$  and  $(B_t)_{t \in [0, T]}$  is a standard Brownian motion,  $(N_t)_{t \in [0, T]}$  is a Poisson process such that  $N_t \sim \mathcal{P}(\lambda t)$  and  $J := (J_k)_{k \geq 1}$  is a i.i.d sequence of random variable with probability distribution  $F$  and  $\mathbb{E}[J] = k < \infty$  [18].

**Proposition 2.6.** [2] *Let  $X := (X_t)_{t \in [0, T]}$  be a Lévy process. Then, for  $t \in [0, T]$  we can rewrite,  $\forall n \geq 1$  and  $0 = t_0 < t_1 < \dots < t_n = t$  (for simplicity we take  $t_i - t_{i-1} = \frac{T}{n}$ ), the Lévy process as*

$$X_t = (X_t - X_{t_{n-1}}) + (X_{t_{n-1}} - X_{t_{n-2}}) + \dots + (X_{t_1} - X_{t_0}).$$

Hence, by following the definition infinite divisibility of random variable, we can find, for all  $n > 1$ , a random variable  $Y^{(n)}$  with law  $\mu^{(n)}$  such that  $X_t$  has the same law as  $\sum_{i=1}^n Y^{(n)} \stackrel{Law}{=} X_{t_1}$  where  $Y_i^{(n)}$  are i.i.d.

The next theorem is the famous Lévy-Khintchine representation which provides a complete characterization of a random variable with an infinitely divisible law.

**Theorem 2.7 (Lévy-Khintchine Representation [1]).** *Let  $\mu \stackrel{Law}{=} X_1$  be an infinitely divisible law on  $\mathbb{R}$ , then the characteristic function or a Fourier transform is given by*

$$\mathbb{E}[e^{iuX}] = \exp \left( iub - \frac{1}{2} \sigma^2 u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux \mathbf{1}_{|x| < 1}) \nu(dx) \right), \quad (2.1)$$

where  $u \in \mathbb{R}$  is called the drift term,  $\sigma \in \mathbb{R}_+$  is the Gaussian and  $\nu$  is a positive measure, called the Lévy measure, which satisfies :  $\nu(\{0\}) = 0$  and  $\int (1 \wedge x^2) \nu(dx) < \infty$ . The triplet  $(b, \sigma^2, \nu)$  is called the characteristic triplet and the exponent

$$\psi(u) = iub - \frac{1}{2} \sigma^2 u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux \mathbf{1}_{|x| < 1}) \nu(dx)$$

is called the characteristic exponent. This representation is unique and the reverse is true. (i.e. if  $\mu$  has a characteristic exponent of the form  $\psi$ ) then  $\mu$  is an infinitely divisible [2].

Therefore, if  $X$  is a Lévy process and we assume that  $\mu \stackrel{Law}{=} X_1$ , then we can see that  $\forall n > 1$ ,

$$\mathbb{E}[e^{iuX_n}] = (\mathbb{E}[e^{iuX_1}])^n = \exp(n\psi(u)).$$

By using the fact that  $X_n$  is Infinitely divisible (ID), has independent increment and stationary increment. Then, we can generalize this result  $\forall t > 0$

$$\mathbb{E}[e^{iuX_t}] = e^{t\psi(u)}. \quad (2.2)$$

This implies that we can find the characteristic triplet of Lévy process at any time  $t > 0$ , when find it at  $t = 1$ . Moreover, if  $\nu = 0$ , then our process becomes a drifted Brownian Motion.

**Definition 2.8** (Random Poisson Measure [18]). Consider a set  $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$  such that  $0 \notin \bar{A}$ . Then, we define a random measure of the jumps of the Lévy process  $L := (L_t)_{t \in [0, T]}$  by:

$$\mu^L(\omega; t, A) = \# \{0 \leq s \leq t; \Delta L_s(\omega) \in A\} = \sum_{s \leq t} \mathbf{1}_A(\Delta L_s(\omega)),$$

$\forall t \in [0, T]$ . This measure gives the number of jumps of the process  $L$  of size in  $A$  for  $s \in [0, t]$ . Where  $\Delta L := (\Delta L_t)_{t \in [0, T]}$  is a jump process associated to the Lévy process  $L$ . The properties of the random Poisson measure are the following [18] :

- (i)  $\mu^L(., A)$  has independent increments.
- (ii)  $\mu^L(., A)$  has stationary increments by stationarity of the increment of  $L$ ;

Therefore, the counting process  $\mu^L(., A)$  is a Poisson process where  $\mu^L$  is a random Poisson measure and the intensity of  $\mu^L$  is given by  $\nu(A) = \mathbb{E}[\mu^L(1, A)]$ .

**Definition 2.9** (Lévy Measure [18]). The density of a Counting process seen in the previous definition is Lévy Measure.

$$\nu(A) = \mathbb{E}[\mu^L(1, A)] = \mathbb{E} \left[ \sum_{s \leq 1} \mathbf{1}_A(\Delta L_s(\omega)) \right],$$

where  $\mu^L$  is a Poisson random measure.

**Definition 2.10** (Stochastic integral with respect to the Poisson random measure  $\mu^L$  [1]). Let us consider  $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ ,  $0 \notin \bar{A}$  and  $f : \mathbb{R} \rightarrow \mathbb{R}$ , a Borel measurable function, finite on  $A$ . Then, we have the following integral

$$\int_A f(x) \mu^L(\omega; t, dx) = \int_{\mathbb{R}} f(x) \mathbf{1}_A \mu^L(\omega; t, dx) = \sum_{s \leq t} f(\Delta L_s) \mathbf{1}_A(\Delta L_s(\omega)), \quad (2.3)$$

where  $\int_A f(x) \mu^L(t, dx)$  is a real-valued random variable and generate a càdlàg stochastic process.

**Theorem 2.11.** [1] *Let us consider the stochastic process*

$\left( \int_0^t \int_A f(x) \mu^L(ds, dx) \right)_{t \in [0, T]}$ , *a set*  $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ ,  $0 \notin \bar{A}$  *and*  $f : \mathbb{R} \rightarrow \mathbb{R}$  *a Borel measurable function and finite on*  $A$ . *Then :*

- (i)  $\left( \int_0^t \int_A f(x) \mu^L(ds, dx) \right)_{t \in [0, T]}$  *is a compound Poisson process with the following characteristic*

$$\mathbb{E} \left[ \exp \left( iu \int_0^t \int_A f(x) \mu^L(ds, dx) \right) \right] = \exp \left( t \int_A \left( e^{iuf(x)} - 1 \right) \nu(dx) \right); \quad (2.4)$$

- (ii) *If*  $f$  *is integrable in*  $A$ , *then*

$$\mathbb{E} \left[ \int_0^t \int_A f(x) \mu^L(ds, dx) \right] = t \int_A f(x) \nu(dx); \quad (2.5)$$

- (iii) *If*  $f$  *is square integrable in*  $A$ , *then*

$$\text{Var} \left( \left| \int_0^t \int_A f(x) \mu^L(ds, dx) \right| \right) = t \int_A |f(x)|^2 \nu(dx). \quad (2.6)$$

*Proof.* For the proof see theorem 2.3.7. in [1].  $\square$

**Theorem 2.12** (Lévy Itô Decomposition [18]). *Consider a characteristic triplet*  $(b, c, \nu)$  *of a Lévy process*  $L := (L_t)_{t \in [0, T]}$ . *Then there exist a probability space*  $(\Omega, \mathcal{F}, \mathbb{P})$  *such that*

$$L = L^{(1)} + L^{(2)} + L^{(3)} + L^{(4)},$$

where the four Lévy processes are independent and  $L^{(1)}$  is a drift process,  $L^{(2)}$  is a Brownian Motion,  $L^{(3)}$  is a compound Poisson process (with jumps more than 1) and  $L^{(4)}$  is a square integrable (pure jump) martingale (with jumps less than 1 on each finite time interval).

Therefore,  $L$  has the following characteristic exponent :

$$\psi(u) = iub - \frac{u^2 c}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux \mathbf{1}_{|x| < 1}) \nu(dx),$$

with  $u, b \in \mathbb{R}$ ,  $c \in \mathbb{R}_+$  and  $\nu$  satisfies  $\nu(\{0\}) = 0$  and  $\int_{\mathbb{R}} (1 \wedge x^2) \nu(dx) < \infty$ .

*Proof.* This proof can be split into two, for the structure of paths of Lévy process and the other one about the Lévy exponent (for the proof see theorem 2.4.11 and 2.4.15 in [1]).  $\square$

We can also give some steps to show that the second part of the proof. Let the Lévy exponent  $\psi$  with a characteristic triplet  $(b, c, \nu)$  be given,  $\forall u \in \mathbb{R}$ , by

$$\begin{aligned} \psi(u) &= \psi^{(1)}(u) + \psi^{(2)}(u) + \psi^{(*)}(u) = iub - \frac{u^2 c}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux \mathbf{1}_{|x| < 1}) \nu(dx) \\ &= iub - \frac{u^2 c}{2} + \int_{|x| \geq 1} (e^{iux} - 1) \nu(dx) + \int_{|x| < 1} (e^{iux} - 1 - iux) \nu(dx). \end{aligned}$$

We can see that  $\psi^{(*)}$  can be written as  $\psi^{(3)} + \psi^{(4)}$ , then we have

$$\psi(u) = \psi^{(1)}(u) + \psi^{(2)}(u) + \psi^{(3)}(u) + \psi^{(4)}(u),$$

**Definition 2.13.** We call a subordinator an a.s increasing (int) Lévy process [18].

In this end of the section we present the stochastic integral with respect to the Lévy Process. We give some important tools for the construction of Lévy integral. We replace the Poisson random measure  $\mu^L$  by the new notation  $N$  for which the intensity is  $\nu$  and the associated compensated Poisson process is  $\tilde{N}$ .

Let  $M$  to be a Hilbert space,  $E = M - \{0\}$  and  $\mathcal{P}_2(T, E)$  be a space of predictable mappings  $F : [0, T] \times E \times \Omega \longrightarrow \mathbb{R}$  for which  $\mathbb{P} \left( \int_0^T \int_E |F(t, x)|^2 \nu(dx) dt \right) < \infty$ .

**Definition 2.14.** [1] Let  $H$  and  $K$  predictable mappings taking values in  $\mathcal{P}_2(T, E)$  and  $Y := (Y(t), t \geq 0)$  be a  $\mathbb{R}$ -valued stochastic process,  $Y$  is said to be Lévy stochastic integral, or Lévy integral, if it can be written,  $\forall t \geq 0$ , in the form:

$$\begin{aligned} Y(t) = & Y(0) + \int_0^t G(s) ds + \int_0^t F(s) dB(s) + \int_0^t \int_{|x| < 1} H(s, x) \tilde{N}(ds, dx) \\ & + \int_0^t \int_{|x| \geq 1} K(s, x) N(ds, dx), \end{aligned} \tag{2.7}$$

$\forall t \in [0, T]$  we have  $F \in \mathcal{P}_2(T)$  and  $G \in L^1[0, T]$ ; Where  $Y^0$  be  $\mathcal{F}_0$ -measurable. Then  $Y$  is adapted process since  $\int_0^t \int_{|x| \geq 1} K(s, x) N(ds, dx)$  has a càdlàg path.

**Proposition 2.15.** Let a one dimensional Lévy integral  $Y$  as defining in equation (2.7). Then,  $Y$  is martingale if

$$G(s) + \int_0^t \int_{|x| \geq 1} K(s, x) \nu(dx) = 0.$$

*Proof.* Let a given predictable mapping  $K$ . Then, we have the following compensated Poisson process by following theorem 2.11

$$\begin{aligned} \int_0^t \int_{|x| \geq 1} K(s, x) \tilde{N}(ds, dx) &= \underbrace{\int_0^t \int_{|x| \geq 1} K(s, x) N(ds, dx)}_{(*)} \\ &\quad - \int_0^t \int_{|x| \geq 1} K(s, x) \nu(dx) ds. \end{aligned} \quad (2.8)$$

By replacing the expression of  $(*)$  in equation (2.7), we obtain the following:

$$\begin{aligned} Y(t) = & Y(0) + \int_0^t \left( G(s) + \int_0^t \int_{|x| \geq 1} K(s, x) \nu(dx) \right) ds + \int_0^t F(s) dB(s) \\ & + \int_0^t \int_{|x| < 1} H(s, x) \tilde{N}(ds, dx) + \int_0^t \int_{|x| \geq 1} K(s, x) \tilde{N}(ds, dx). \end{aligned} \quad (2.9)$$

□

*Remark 2.16.*  $Y$  is a semimartingale (see page 234 in [1]).

Let  $X := (X(t), t \geq 0)$  be a  $\mathbb{R}$ -valued Lévy process and  $f \in L^2(\mathbb{R}^+)$ . Let us consider the Lévy Integral  $Y = (Y(t), t \geq 0)$  where :

$$Y(t) = \int_0^t f(s) dX(s), \quad (2.10)$$

then  $Y$  has independent increments (see Lemma 4.3.12 in [1]).

Let us take for simplification  $f \in L^2(\mathbb{R})$  and shift  $s \rightarrow s - t$  such that  $f(s - t) \in L^2(\mathbb{R})$  where we assume that  $f$  is a càdlàg. Then, the moving-average process  $Z = (Z(t), t \geq 0)$  is given by

$$Z(t) = \int_{-\infty}^t f(s - t) dX(s). \quad (2.11)$$

The moving-average process  $Z = (Z(t), t \geq 0)$  is stationary (see [1], theorem 4.3.16).

The Ornstein-Uhlenbeck process driven by a Lévy process is a special case of the moving-average process by taking  $f(s) = e^{\lambda s}$  for  $s \leq 0$  where we fix  $\lambda > 0$ , we have

$$Z(t) = \int_{-\infty}^t f(s - t) dX(s) = e^{-\lambda t} Z(0) + \int_0^t e^{-\lambda(t-s)} dX(s), \quad \forall t \geq 0. \quad (2.12)$$

Since the Ornstein-Uhlenbeck process can be written as

$$Z(t) = e^{-\lambda t} Z(0) + Z_1(t).$$

Then  $Z$  is self decomposable since  $e^{-\lambda t} Z(0)$  and  $Z_1(t)$  are independent by independent increment of Lévy integral and  $Z(t) \stackrel{Law}{=} Z(0)$ . We have  $\forall t > 0$

$$\phi(u) = \phi(e^{-\lambda t} u) \phi_t(u), \quad (2.13)$$

where  $\phi_t(u)$  is the characteristic function of  $Z_1(t)$  [1].

*Remark 2.17.* When  $X$  is used to generate a self decomposable random variable in this way, the Lévy process  $X$  is said to be a BDLP (i.e. Background-Driving Lévy Process).

**Theorem 2.18** (See theorem 4.3.17 in [1]). *We have the following are equivalent:*

- (i).  $Z$  is a self-decomposable random variable;
- (ii).  $\int_{|x|>1} \log(1+|x|)\nu(dx) < \infty$ , where  $\nu$  is the Lévy measure of  $X$ ;
- (iii).  $Z$  can be represented as  $Z(0)$  in a stationary Ornstein-Uhlenbeck process  $(Z(t), t \geq 0)$ .

### 3. Superposed Ornstein-Uhlenbeck Processes

In this section we show and give some tools for the construction of the superposition of Ornstein-Uhlenbeck processes introduced by Barndorff-Nielsen [15]. The OU and supOU processes give a way to construct LRD (Long-Range Dependence) process with a given marginal distribution since OU processes is self-decomposable [13].

To explain the present value of a given series  $X_t$  by function of  $p$  past values,  $X_{t-1}, \dots, X_{t-p}$ , we use the concept of Auto Regressive Process of order  $p$  (AR( $p$ )).

**Definition 3.1.** [13] An autoregressive process of order 1 or AR(1) is a process of the form

$$X_t = cX_{t-1} + Z_t, \quad (3.1)$$

where  $Z_t$  is an i.i.d random variable independent of  $X_{t-1}$ .

*Remark 3.2.* when  $c \in (0, 1)$  a AR(1) is self-decomposable. Hence, in this condition, OU process is a AR(1) [13].

**Definition 3.3** (Cumulant Function [5]). Let  $X$  be a random variable. We call a cumulant function of  $X$ , the function defined as follows

$$\kappa_X(\zeta) = \mathbb{C} \{ \zeta \dagger X \} = \log \mathbb{E} \{ e^{i\zeta X} \}. \quad (3.2)$$

We define the  $m^{th}$  cumulant function of  $X$  by ([5]):

$$\kappa_X^{(m)} = (-i)^m \frac{d^m}{d\zeta^m} \kappa_X(\zeta) \Big|_{\zeta=0}. \quad (3.3)$$

We denote the cumulant function of a stochastic process  $X(t)$  or  $X_t$  by  $\kappa_{X(t)}(\zeta)$  and its  $m^{th}$  cumulant function by  $\kappa_{X(t)}^{(m)}(\zeta)$ .

*Remark 3.4.* For any infinitely divisible random variable  $X$  corresponds a Lévy process  $z$  such that  $X$  has the same law as  $z(1)$  (See Page 5 in [15]). We say that  $z$  is generated by  $X$ .

**Proposition 3.5.** [15] *Let  $(x(t), t \in T)$  be an infinitely divisible stochastic process with  $T$  an index set. Then such process generates a generalized Lévy process  $z = (z(s, t) : s \geq 0, t \in T)$  by prescription:*

$$\mathbb{C} \{ \zeta_1, \dots, \zeta_m \dagger z(s, t_1), \dots, \dots, z(s, t_m) \} = s \mathbb{C} \{ \zeta_1, \dots, \zeta_m \dagger x(s, t_1), \dots, x(s, t_m) \}, \quad (3.4)$$

for all finite dimensional laws.



Using the two following notations for the two cumulant functions as :

$$\kappa(\zeta) = C\{\zeta \sharp x\} \quad \text{and} \quad \hat{\kappa}(\zeta) = C\{\zeta \sharp z(1)\},$$

and using the relation  $C\{\zeta \sharp dz(s)\} = C\{\zeta \sharp z(1)\} ds$ , we want to show the link between the cumulant function of an infinitely divisible  $x$  and the law of Lévy process  $z(1)$ . Using the following definition

$$z(1) = \int_0^\infty e^{-s} dz(s), \quad (3.5)$$

we have,

$$\kappa(\zeta) = \log \mathbb{E} \left[ e^{i\zeta z(1)} \right] = \log \mathbb{E} \left[ e^{i\zeta \int_0^\infty e^{-s} dz(s)} \right] = \int_0^\infty \hat{\kappa}(\zeta e^{-s}) ds.$$

Where  $\hat{\kappa}(\zeta) = \zeta \kappa'(\zeta)$ .

**Proposition 3.6.** *We say that a random variable  $x$  is self-decomposable if its representation has the same form as in equation (3.5) [15].*

### 3.1. Independent Scattered Random Measure.

**Definition 3.7.** [15] Let  $\Omega$  be a Borel subset of  $\mathbb{R}^d$ .  $\mathcal{R}$  is said to be a  $\sigma$ -ring of  $\Omega$  if the following properties holds :

- (i) If a sequence  $(A_n)_{n \in \mathbb{N}} \in \mathcal{R}$ , then  $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{R}$ ;
- (ii) Let  $A$  and  $B \in \mathcal{R}$  with  $A \subset B$ , then  $B \setminus A \in \mathcal{R}$ .

**Definition 3.8.** [15] Let a random variable be defined as  $z = \{z(A); A \in \mathcal{R}\}$ . Then, a collection of  $z$  defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  is called an *independently scattered random measure* if for every sequence  $\{A_n\}$  of disjoint sets in  $\mathcal{R}$ , the random variables  $z(A_n), n = 1, 2, \dots$ , are independent and if

$$z\left(\bigcup_{n=1}^\infty A_n\right) = \sum_{n=1}^\infty z(A_n), \quad \text{a.s.},$$

Taking the case where  $z$  is ID, then each  $A \in \mathcal{R}$ ,  $z(A)$  is ID random variable for which a cumulant function is written as follows (the current definition can be found in [15]):

$$C\{\zeta \sharp z(A)\} = i\zeta m_0(A) - \frac{1}{2}\zeta^2 m_1(A) + \int_{\mathbb{R}} (e^{i\zeta x} - 1 - i\zeta \mathbb{1}_{[-1,1]}(x)) Q(A, dx), \quad (3.6)$$

where  $m_0$  is a signed measure,  $m_1$  is a positive measure and for a fixed  $A$ ,  $Q(A, dx)$  is a measure on  $\mathcal{B}(\mathbb{R})$  with  $Q(A, \{0\}) = 0$  such that  $\int_{\mathbb{R}} \min(1, x^2) Q(A, dx) < \infty$ . In this special case  $z$  is said to have the Lévy characteristics  $(m_0, m_1, Q)$  and  $Q$  is called the generalized Lévy measure.

**Proposition 3.9.** [15] *Let an integrable function  $f$  on  $\Omega$  with respect to the random measure  $z$ , then the cumulant function of  $\int_A f dz$  is:*

$$C\left\{\zeta \sharp \int_A f dz\right\} = \int_A \kappa(\zeta f(\omega)) M(d\omega),$$

where  $M$  is a measure on  $\mathcal{R}$ .

*Proof.* For the proof see proposition 2.1. in [15].  $\square$

In the following  $\Omega = \mathbb{R} \times \mathbb{R}_+$ ,  $\omega = (s, \xi)$  and  $z(d\omega) = z(d\xi, ds)$ .

**Definition 3.10.** [5] Let  $\kappa$  be the cumulant function of some decomposable law,  $(a, b, \nu)$  be the characteristic triplet of the associated BDLP with a cumulant function  $\kappa$  and let  $\pi$  be a probability measure on  $\mathbb{R}_+$ . Then, the process  $X = (X(t), t \in \mathbb{R})$  is defined by:

$$X(t) = \int_{\mathbb{R}_+} \int_{\mathbb{R}} e^{-\xi t + s} \mathbb{1}_{[0, \infty)}(\xi t - s) z(d\xi, ds), \quad (3.7)$$

is strictly stationary process. Moreover, for  $t_1 < \dots < t_m$ , the joint cumulant function of  $(X(t_1), \dots, X(t_m))$  exists and has the form:

$$\begin{aligned} C\{\zeta_1, \dots, \zeta_m \dagger X(t_1), \dots, X(t_m)\} \\ = \int_{\mathbb{R}_+} \int_{\mathbb{R}} \left( \sum_{j=1}^m \mathbb{1}_{[0, \infty)}(\xi t_j - s) \zeta_j e^{-\xi t_j + s} \right) ds \pi(d\xi), \end{aligned} \quad (3.8)$$

where we assume its correlation function to be of the form

$$r(\tau) = \int_{\mathbb{R}_+} e^{-\tau \xi} \pi(d\xi), \quad \tau \geq 0. \quad (3.9)$$

Hence,  $X = (X(t), t \in [0, T])$  is called a superposed Ornstein-Uhlenbeck processes. In particular

$$X(t) = \sum_{k=1}^m X^{(k)}(t), \quad (3.10)$$

is a superposed Ornstein-Uhlenbeck processes where for  $\{X^{(k)}(t)\}_{k \geq 1}$  each  $X^{(k)}$  are independents Ornstein-Uhlenbeck processes.

**3.2. The Barndorff-Nielsen and Shephard Model.** In 2003, Barndorff - Nielsen and Shephard [16] published a paper on Realized power variation which is a sum of the absolute power of increments of a process of type *semimartingale*, in particular for stochastic volatility model. The result of the later paper is helpful for volatility model using high-frequency information [16].

Following their two conditions, Barndorff-Nielsen and Shephard proposed a volatility model which can perform to a short term aspect. This volatility model is driven by a BDLP for which the jumps are positive (as a subordinator). That process defined as a following is a Ornstein-Uhlenbeck process driven by a Lévy process,

$$d\sigma^2(t) = -\lambda \sigma^2(t) dt + dZ(\lambda t). \quad (3.11)$$

$\sigma^2$  helps us to be sure on a positivity of the volatility  $\forall t \geq 0$  adding the fact that we have a subordinator. This process is called *Non-Gaussian-Gaussian* Volatility process [7].

Then, for the Barndorff-Nielsen and Shephard model the volatility process will be a superposed Ornstein-Uhlenbeck processes defined as

$$\sigma^2(t) = \sum_{k=1}^m w_k X^{(k)}(t), \quad (3.12)$$

where

$$dX^{(k)}(t) = -\lambda_k X^{(k)}(t)dt + dZ^{(k)}(\lambda_k t). \quad (3.13)$$

We consider  $w_k \geq 0$  and  $Z^{(k)}(\lambda_k t)$  are independent but not necessary i.i.d. subordinator. Following the BN-S conditions on a mean and volatility, we can replace in the Black Scholes equation the expression of a mean and volatility by  $\alpha(t) = (\mu + \sigma^2(t))$  and  $\sigma(t)$ . Hence, we get the equation of the Barndorff-Nielsen and Shephard model,

$$dS(t) = (\mu(t) + \sigma^2(t))S(t)dt + \sigma(t)S(t)dW(t). \quad (3.14)$$

#### 4. Simulation and Application of a Superposed Ornstein-Uhlenbeck Processes

In this last section, we present some discretization and simulation of Black-Scholes equation, Barndorff-Nielsen and Shephard Model with Gaussian OU process and with supOU process. We provide tools on the estimation of the parameters of SDEs. The main goal of this section is to show which of these three models provide a fit close to a real financial data. We use financial data for gold prices between 1950 January and 1999 December from datahub.io<sup>2</sup>.

**4.1. Euler Schemes.** In this part we present a useful discretization of Euler for SDEs. The study of stability and convergence of this scheme can be found in [10, 11].

Consider a SDE of the form

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \quad \forall t \in [0, T]. \quad (4.1)$$

We can simulate the discretized version of the SDE in particular the process  $\{\hat{X}_{jh}\}$  for  $j = 0, 1, 2, \dots, m$  where,  $m$  is the number of time steps given by  $m = \lfloor T/h \rfloor$  with  $h$  the path of the discretization. The Euler scheme of SDE (4.1) is given by :

$$\hat{X}_{jh} = \hat{X}_{(j-1)h} + \mu((j-1)h, \hat{X}_{(j-1)h})h + \sigma((j-1)h, \hat{X}_{(j-1)h})\sqrt{h}Z_j, \quad (4.2)$$

where all  $Z_j$  are i.i.d  $\mathcal{N}(0, 1)$  and  $Z_0 = 0$ . This discretization is applied to our model.

**4.2. Random walk approximation for Lévy process.** Since we are not working on a particular case of the distribution of the Levy process, we chose the Random walk approximation of the Lévy process.

Generally, it is very difficult to simulate a Lévy process, in particular when the Lévy measure  $\nu$  of the triplet characteristic  $(a, b, \nu)$  is infinite [12]. Then, is sufficient to simulate a jump-diffusion process when  $\nu$  is finite [11].

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<sup>2</sup><https://datahub.io/core/gold-prices>

Because of the stationary independent increments, the problem of simulating a discrete skeleton  $\{Z_n\}$  of a Lévy process is equivalent to the problem of a random variable generation from a specific infinitely divisible distribution, where  $Z_n \stackrel{Law}{=} Z(t_n)$  with  $t_n = nh$  [11]. Suppose  $(Z(t); t \geq 0)$  is a Lévy process with characteristic triplet  $(a, 0, \nu)$ , a subordinator, then for a fixed time in  $[0, T]$  with  $m = \lfloor T/h \rfloor$ , generate the increments  $\Delta Z_j = Z_j - Z_{j-1} = Z_{jh} - Z_{(j-1)h}$  as i.i.d random variables with common distribution (replacing  $h$  by  $\Delta$ )  $\mathbb{P}_\Delta(A) = \mathbb{P}(Z(\Delta) \in A), j = 1, 2, \dots, n-1$ . Let

$$Z^\Delta(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \Delta \\ \sum_{k=1}^j \Delta Z_k & \text{if } j\Delta \leq t \leq (j+1)\Delta \end{cases} \quad (4.3)$$

The specific expression of  $\Delta Z_i$  is given on Ernest Jum's paper [11] page 18 Example 2.2.1 .

Then, Let the SDE driven by a Levy process be

$$d\sigma(t) = -\lambda\sigma(t)dt + dX(\lambda t), \quad (4.4)$$

where  $X$  is a subordinator referred to a BDLP. By using the discretization given by (4.2) and (4.3) , we get

$$\hat{\sigma}(jh) = \hat{\sigma}((j-1)h) - \lambda\hat{\sigma}((j-1)h)h + \Delta X(j).$$

For simplicity we take  $X(t)$  by using the stationarity of  $X$ . We can observe that the main difference in the trajectory result from the fact that the OU DBLP has some jumps which look like Lévy process path. The value of  $\alpha$ , generally in  $(0, 2)$ , change the form the trajectory of the Lévy process which change as Brownian Motion, compound Poisson process and Lévy process.

Then, let the process be define as :

$$\sigma^2(t) = \sum_{k=1}^m \omega_j X^{(k)}(t), \quad m \in \mathbb{N}, \quad (4.5)$$

to be a supOU. Where  $\sum_{k=1}^m \omega_k = 1$ . For simplicity, in the simulation, we take  $\omega_k$  to be the same. And, we can observe that the supOU process looks like the average of given OU processes. It reduces the size of jumps and spreads throughout  $\sigma(0)$ . Hence, in this part we fit the three previous financial models, the Black-Scholes model (BS), the Barndorff-Nielsen and Shephard model driven by a Gaussian OU process (BN-S) and the Barndorff-Nielsen and Shephard model with supOU (BN-S with supOU). The main idea is to show how the BN-S with supOU provide a better fit than the BS model.

### 4.3. Calibration of parameters.

(1) For the BS Model, the log-return and the volatility are as follows [8]:

$$\mu(t) = \log \left( \frac{S(t)}{S(t-1)} \right) \quad \text{and} \quad \sigma = \sqrt{\frac{1}{n-1} \sum_{t=1}^n (\mu(t) - \bar{\mu})^2}, \quad (4.6)$$

where  $n$  is the number of observation of the price  $S(t)$ .

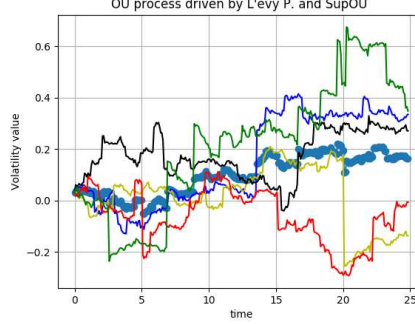


FIGURE 1. superposed Ornstein-Uhlenbeck processes. Here :  
 $T = 25$ ,  $h = 0.01$ ,  $\lambda = 0.2$ ,  $\sigma(0) = 0.05$ ,  $\alpha = 1.4$  and  $m = 6$ .

- (2) For the BN-S model, the log-return will be the same as equation (4.6) but for the volatility, we will fix  $\sigma(0) = \sigma$  where  $(\sigma(t), t \in [0, T])$  is solution of equation (4.4). Let the following scheme :

$$\hat{\sigma}(t) = \hat{\sigma}(t-1) - \lambda \hat{\sigma}(t-1)h + \sqrt{h}Z_j,$$

by posing  $1 - \lambda = \beta$ , we get

$$\hat{\sigma}(t) = \beta \hat{\sigma}(t-1) + Z_j,$$

which is a AR(1).. The value of the constant  $\beta$  is obtained by fitting an AR(1) model on the monthly volatility data.

- (3) For the BN-S with supOU the log-return is the same as equation(4.6).  $\sigma(0) = \sigma$  and each  $(\sigma^{(k)}(t), t \in [0, T])$ ,  $k \in [1, m]$ , are solution of equation (4.4) where we will consider all the  $\lambda_k$  to be the same.

**4.4. Presentation of the Results and discussion.** As said earlier, our data concern gold prices from January 1950 through December 1999. The total number of observation is 600. The parameters are calculated using equations (4.6).

In the Fig 2, we can observe that the trajectory given by the BN-S with supOU is the only one who is close than the real data. This figure show us that, the trajectory of the BS model fluctuated for a small amplitude value due to the constant volatility. This observation shows that the BS model does not perform when the data has hight frequency of informations. As said before, the fact that the volatility is a stochastic process helps the BN-S and the BN-S with supOU to perform. Hence, this kind of volatility process given by supOU is very useful for high frequency information and helps trajectory of price process to jump in value on the same time (somewhere) with the real data. This is where the volatility of BN-S model (with Gaussian OU volatility process) fails.

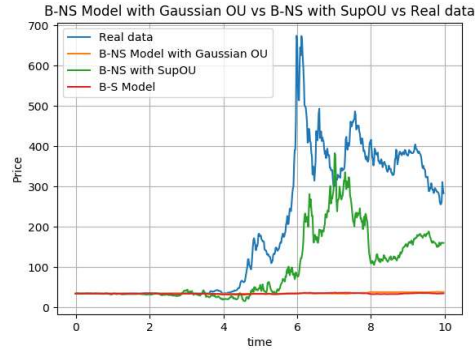


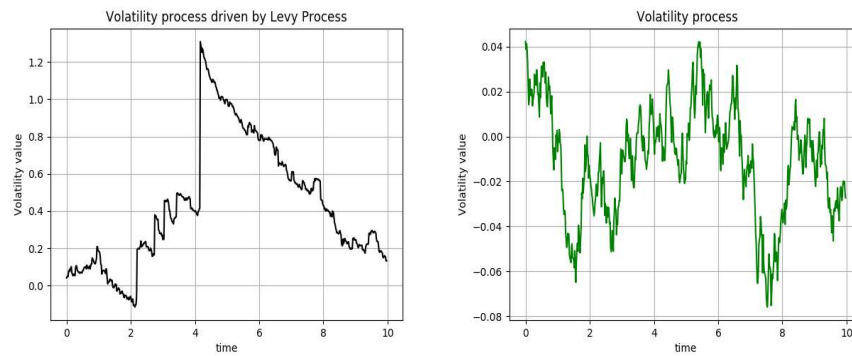
FIGURE 2. BN-S model with Gaussian OU vs BN-S model with supOU vs BS model vs Real data. Here :  $\bar{\mu} = 0.0035$ ,  $\lambda = 0.34$ ,  $\sigma = 0.0424$ ,  $m = 6$ ,  $\alpha = 1.4$  and  $S(0) = 34.73$ .

	mean	Standard deviation
Estimated value	194,59	170,07

FIGURE 3. Parameter estimation of gold prices.

	$\bar{\mu}$	$\sigma(0) = \sigma$	$\omega_K$	$\lambda_K$
Estimated value	0,0035	0,0427	0,16667	0,34

FIGURE 4. Parameter estimation for return of gold prices.



## 5. Conclusion

To model the price of a financial asset, we must always take into account the disturbances of the market. This disruption is reflected in the volatility of the asset

price. In this work, our objective is to find a model that can overcome the flaws of the Black-Scholes model where the value of the volatility does not change. On this basis, we investigate the superposition of Ornstein-Uhlenbeck processes, which has jumps trajectory, to model the volatility of asset prices and we chose the Barndorff-Nielsen and Shephard model to model asset price. Using real data for the prices of gold between 1950 and 1999, we observed that the trajectory of superposition of Ornstein-Uhlenbeck processes is similar to that of a Lévy process. This trajectory makes the trajectory of SDE of the Barndorff-Nielsen and Shephard model close to that of the real data, unlike the Gaussian Ornstein-Uhlenbeck which has the tendency of a Brownian motion.

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