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MARKOV SEMIGROUPS AND GROUPS OF OPERATORS

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ABSTRACT. We consider different realizations of the operators $L_{\theta,a}\ u(x) := x^{2a}u''(x) + (ax^{2a-1}+\theta x^a)u'(x)$, $\theta\in\mathbb{R}$, $a\in\mathbb{R}$, acting on suitable spaces of real valued continuous functions. The main results deal with the existence of Feller semigroups generated by $L_{\theta,a}$ and the representation $L_{\theta,a} = G_a^2 + \theta G_a$, where $G_a u = x^a u'$, $0\le a\le 1$, generates a (not necessarily strongly continuous) group. Explicit formulas of the generated semigroups are also deduced.

1. Introduction

Let us denote by $C(\overline{J})$ the space of all real valued continuous functions on an interval J, having finite limits at all endpoints not included in J, equipped with the sup-norm.

We are interested in the operators

$$L_{\theta,a}\ u(x) \ :=\ x^{2a}u''(x) \ +\ (ax^{2a-1}+\theta x^a)u'(x),$$

where $\theta\in\mathbb{R}$, $a\in\mathbb{R}$.

In [7], using Feller classification of the boundary points we showed that for *any* $\theta\in\mathbb{R}, a\in\mathbb{R}$, the operator $L_{\theta,a}$ generates a Feller semigroup on $C[0,+\infty]$. Here, in addition to analogous generation results in different spaces of continuous functions, we obtain an explicit representation of the semigroup generated by $L_{\theta,a}$ for suitable a. Indeed, if $0\le a\le 1$ and $G_a u := x^a u'$, then the operator $L_{\theta,a}$ can be represented as

$$L_{\theta,a}u = G_a^2 u + \theta G_a u,$$

where G_a generates a (not necessarily strongly continuous) group. Thus a variant of Romanov's formula applies and the results follow. For the connections with the Black-Merton-Scholes equation see [8].

In the following, for any Banach space X, $C(\mathbb{R},X)$ will denote the Banach space of all X-valued continuous functions defined in $\mathbb{R}$ and $\mathcal{L}_{\mathbb{K}}(X)$ the Banach algebra of all linear bounded operators on a Banach space X over $\mathbb{K}$, for $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$.

2. Feller semigroups, analytic semigroups and cosine functions

This section provides a brief description of the basic definitions and results about Feller semigroups, analytic semigroups and cosine functions, which form a functional analytic background for our results.

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Let us consider a locally compact, separable, metric space (K, ρ) and define $K_\partial = K \cup \partial$, where ∂ is the point at infinity, if K is not compact. Hence K_∂ is compact and ∂ is some point disjoint from K if K is compact. Define $C(K_\partial)$ to be the space of all real-valued continuous functions on K_∂. The space $C(K_\partial)$ is a Banach space with the maximum norm $\|f\| = \sup_{x\in K_\partial} |f(x)|$. Observe that for $K = [0, +\infty)$, we have $C(K_\partial) = C[0, +\infty]$. Define the subspace $C_0(K)$ as follows

$$C_0(K) = \{f \in C(K_\partial) \,:\, f(\partial) = 0\}.$$

The space $C_0(K)$ is a closed subspace of $C(K_\partial)$, hence it is a Banach space. Note that $C_0(K)$ can be identified with $C(K)$ if K is compact.

Let us recall the notions of C_0-semigroup and of Feller semigroup.

Definition 2.1. A family $(T(t))_{t\geq 0}$, $T(t) \in \mathcal{L}_{\mathbb{K}}(X)$ is called a C_0-semigroup on X if it satisfies the following conditions:

(i): $T(t+s) \;=\; T(t)\,T(s), \quad t, s \geq 0;\ T(0) \;=\; I;$

(ii): $(T(t))_{t\geq 0}$ is strongly continuous in t, i.e.

$$\lim_{s\downarrow 0} \|T(t+s)\,f - T(t)\,f\| \;=\; 0, \quad f \in X,\, t \geq 0.$$

A family $(T(t))_{t\geq 0}$, $T(t) \in \mathcal{L}_{\mathbb{R}}(C(K_\partial))$ is a Feller semigroup on $C(K_\partial)$ if it satisfies (i) and (ii), and, in addition, the following property:

$$(f \in C(K_\partial),\, 0 \leq f \leq 1 \text{ on } K_\partial) \;\Rightarrow\; (0 \leq T(t)\,f \leq 1,\, t \geq 0, \text{ on } K_\partial,\, T(t)1 = 1,\, t \geq 0).$$

Feller semigroups can be related to particular classes of Markov transition functions, i.e. the so-called uniformly stochastically continuous transition functions, defined as follows.

Definition 2.2. A transition probability function P_t, $t \geq 0$, on K is said to be *uniformly stochastically continuous* on K if the following condition is satisfied: For each $\varepsilon > 0$ and each compact $E \subset K$, we have

$$\lim_{t\downarrow 0} \sup_{x\in E} \left[1 \,-\, P_t(x,\, U_\varepsilon(x))\right] \;=\; 0,$$

where $U_\varepsilon(x) = \{y \in K \,:\, \rho(x, y) < \varepsilon\}$ is an ε-neighborhood of x.

More precisely, the following result holds (see e.g. [13, Theorem 9.2.3]).

Theorem 2.3. *The following statements are equivalent:*

(a): $(P_t)_{t\geq 0}$ *is a uniformly stochastically continuous C_0-transition function on K and satisfies the condition*

(L) For each $s > 0$ and each compact $E \subset K$, it follows that

$$\lim_{x\to\partial} \sup_{0\leq t\leq s} P_t(x, E) \;=\; 0;$$

(b): *The family of operators $(T_t)_{t\geq 0}$, defined by*

$$T(t)\,f(x) \;=\; \int_K P_t(x, dy)\,f(y), \quad f \in C_0(K),$$

is a Feller semigroup on $C_0(K)$.

Notice that any Feller semigroup (and hence any corresponding family of uniformly stochastically continuous C_0 transition functions) is uniquely associated to a suitable operator $(A, D(A))$ called the generator of the Feller semigroup, defined in the following way: the domain of A is the subspace $D(A)$ given by

$$D(A) = \left\{ u \in C_0(K) : \text{there exists} \quad \lim_{t\downarrow 0} \frac{T(t)\, u - u}{t} \in C_0(K) \right\}$$

and, for any $u \in D(A)$,

$$Au \ = \ \lim_{t\downarrow 0} \frac{T(t)\, u - u}{t}.$$

According to the Lumer-Phillips version of the Hille-Yosida theorem (see e.g. [6, Theorem 3.3]), the simplest method to show that a closed, densely defined, linear operator $(A, D(A))$ on $C_0(K)$ generates a C_0-contraction semigroup is to check that the operator $(A, D(A))$ is dissipative and satisfies the range condition. The notion of dissipativity relies on the duality map.

Definition 2.4. Let X be a Banach space with dual space X' and $< \cdot, \cdot >$ be the pairing between X and X'. For every $x \in X$, we say that $j(x)$ is the duality map of x if

$$j(x) \ = \ \{x' \in X' : < x, x' > = \|x\|^2 \ = \ \|x'\|^2\}.$$

Notice that Hahn-Banach theorem implies that $j(x) \neq \emptyset$ for any $x \in X$. In particular, for $X = C_0(K)$, if $f \in X$, $f \neq 0$ and δ_{t_0} denotes the evaluation function at t_0, we have:

$$\{\overline{f(t_0)} \cdot \delta_{t_0} \ : \ t_0 \in K, \ |f(t_0)| \ = \ \|f\|\} \subset j(f).$$

Definition 2.5. An operator $(A, D(A))$ on a Banach space X is called dissipative if, for any $f \in D(A)$, there exists $x' \in j(f)$ such that $Re < Af, x' > \leq 0$.

Now the Lumer-Phillips version of the Hille-Yosida theorem reads as follows (see [6, Chapter I, Section 3]).

Theorem 2.6. *Let $(A, D(A))$ be a linear operator on a Banach space X. Then $(A, D(A))$ generates a C_0-contraction semigroup if and only if A is densely defined and m-dissipative (i.e. $(A, D(A))$ is dissipative and $\rho(A) \cap (0, +\infty) \neq \emptyset$).*

Definition 2.7. A C_0-group T on a Banach space X over $\mathbb{K}$ is a family $(T(t))_{t\in\mathbb{R}}$ of elements of $\mathcal{L}_{\mathbb{K}}(X)$ satisfying the conditions of Definition 2.1 but with $\mathbb{R}_+$ replaced by $\mathbb{R}$. The generator $(A, D(A))$ of a C_0-group on X is defined by

$$Af \ = \ \lim_{t\to 0} \frac{T(t)\, f - f}{t},$$

the domain of A being the subspace

$$D(A) = \left\{ f \in X \ : \ \lim_{t\to 0} \frac{T(t)\, f - f}{t} \in X \right\}.$$

Remark 2.8. Note that here the limit is a two-sided one. Moreover, $(A, D(A))$ is the generator of a C_0-group $(T(t))_{t\in\mathbf{R}}$ if and only if $\pm A$ generates a C_0-semigroup $(T_\pm(t))_{t\geq 0}$, where

$$T(t) \ = \ \begin{cases} T_+(t), & t \geq 0 \\ T_-(-t), & t \leq 0. \end{cases}$$

Observe that, for $X = C_0(K)$, the automorphism groups on X can be characterized as follows (see e.g. [11, Propositions 3.8, 3.9]).

Proposition 2.9. *Let $\Phi : \mathbb{R} \times K \longrightarrow K$ be a flow $(\Phi_t)_{t\in\mathbb{R}}$ on K (i.e. $\Phi_t : K \to K$, $\Phi_t(x) = \Phi(t,x)$, is continuous for any $t \in \mathbb{R}$, and $\Phi_0(x) = x$, $x \in K$, $\Phi_s \circ \Phi_t = \Phi_{s+t}$, $s,t \in \mathbb{R}$). Let $(h_t)_{t\in\mathbb{R}}$ be a cocycle of Φ (i.e. $(h_t)_{t\in\mathbb{R}}$ is a family of real-valued bounded continuous functions on K such that $h_0 = 1$ and $h_{t+s} = h_t \cdot (h_s \circ \Phi_t)$, $s,t \in \mathbb{R}$).*
If, for every $x \in K$, the mappings

$$t \mapsto \Phi_t(x), \qquad t \mapsto h_t(x)$$

are continuous, then the operator $T(t)\, f = h_t \cdot (f \circ \Phi_t)$ defines a C_0-group. Conversely, if $(T(t))_{t\in\mathbb{R}}$ is a C_0-group of positive operators on $C_0(K)$, then there exist a continuous flow on K and a continuous cocycle $(h_t)_{t\in\mathbb{R}}$ of Φ such that $T(t)\, f = h_t \cdot (f \circ \Phi_t)$, for any $f \in C_0(K)$, $t \in \mathbb{R}$.

Among all possible C_0-semigroups, the most regular class is the class of analytic semigroups, defined as follows.

Definition 2.10. For $\alpha \in (0, \pi]$ we define the sector $S(\alpha)$ in the complex plane by

$$S(\alpha) \;=\; \{r\, e^{i\theta} \,:\, r > 0,\, \theta \in (-\alpha, \alpha)\}.$$

A C_0-semigroup $(T(t))_{t\geq 0}$ on a (complex) Banach space X is called a bounded analytic semigroup of angle $\alpha \in \left(0, \frac{\pi}{2}\right]$ if $(T(t))_{t>0}$ is the restriction of an analytic function

$$T(\cdot) : S(\alpha) \longrightarrow \mathcal{L}_{\mathbb{C}}(X)$$

satisfying

- ***i):*** $T(z)\, T(z') \;=\; T(z+z'), \quad z, z' \in S(\alpha)$;
- ***ii):*** For each $\alpha_1 \in (0, \alpha)$ the set $\{T(z) : z \in S(\alpha_1)\}$ is uniformly bounded and $\lim_{n\to\infty} T(z_n)\, f = f$ for any null-sequence (z_n) in $S(\alpha_1)$ and every $f \in X$.

We say that A generates an analytic semigroup of angle α if for every $\varepsilon > 0$ with $\alpha - \varepsilon > 0$ there is an $\omega = \omega(\varepsilon)$ such that $A - \omega I$ generates a bounded analytic semigroup of angle $\alpha - \varepsilon$.

Observe that the generators of analytic semigroups can be characterized as follows (see e.g. [5]).

Theorem 2.11. *Let $(A, D(A))$ be a densely defined operator on a Banach space X and $\alpha \in \left(0, \frac{\pi}{2}\right]$. Then $(A, D(A))$ is the generator of an analytic semigroup of angle α if and only if there exists $R > 0$ such that*

$$\lambda \in S\left(\alpha + \frac{\pi}{2}\right), |\lambda| \geq R \quad \textit{implies} \quad \lambda \in \rho(A)$$

and for every $\alpha_1 \in (0, \alpha)$ there exists a constant $M \geq 0$ such that

$$\|R(\lambda, A)\| \leq \frac{M}{|\lambda|}, \qquad \lambda \in S\left(\alpha_1 + \frac{\pi}{2}\right), |\lambda| \geq R.$$

In the case of second order Cauchy problems the corresponding notion of generator is given by means of the definition of generator of a cosine function (see e.g. [6, Chapter II, Section 8]).

Definition 2.12. A (strongly continuous) cosine function on a Banach space X is a family $C = (C(t))_{t\in\mathbb{R}}$ of linear bounded operators on X satisfying

(i): $C(t+s) + C(t-s) = 2C(t)\,C(s) \qquad t, s \in \mathbb{R}$;
(ii): $C(0) = I$;
(iii): $C(\cdot)\,f \in C(\mathbb{R}, X)$, for each $f \in X$.

The generator $(A, D(A))$ of a cosine function C is the operator $A := C''(0)$, with domain

$$D(A) \; := \; \{f \in X \,:\, C(\cdot)\,f \in C^2(\mathbb{R}, X)\}.$$

There are significant relations among generators of C_0-groups, generators of cosine functions and generators of analytic semigroups. We collect some of them in the following theorem (see e.g. [6, Chapter II, Section 8]).

Theorem 2.13. *(i) If $(B, D(B))$ is the generator of a C_0-group $(T(t))_{t\in\mathbb{R}}$ on a Banach space X, then for any $a \in \mathbb{R}$, the operator $(aI + B^2, D(B^2))$ generates a cosine function C_a on X. If $a = 0$, then $(B^2, D(B^2))$ generates a cosine function C_0 given by*

$$C_0(t) \; = \; \frac{[T(t) + T(-t)]}{2}, \qquad t \in \mathbb{R} \;\; (d'Alembert's \quad formula).$$

(ii) Let $(A, D(A))$ be the generator of a cosine function C on a Banach space X. Then $(A, D(A))$ generates a C_0-semigroup $(T(t))_{t\geq 0}$ given by

$$T(t)\,f \; = \; \frac{1}{\sqrt{\pi\, t}} \int_0^{\infty} e^{-\frac{y^2}{4t}}\, C(s)\, f\, ds, \qquad t > 0 \;\; (Romanov's \quad formula)$$

for any $f \in X$. In addition, if X is a complex space, then $(T(t))_{t\geq 0}$ is an analytic semigroup in the right half plane.

3. Feller semigroups and explicit representations in $C_0(\mathbb{R}_+)$

If $J = (r_1, r_2)$ is a real interval, with $-\infty \leq r_1 < r_2 \leq +\infty$, let A be a second order differential operator of the type

$$Au := a(x)u'' + b(x)u',$$

where a and b are real valued continuous functions on J such that $a(x) > 0$ for any $x \in J$. Then we can introduce the Feller classification of the boundary (see e.g. [5, Chapter VI Section 4]) . Let us denote by

$$W(x) \;\; := \;\; \exp\left(-\int_{x_0}^{x} \frac{b(s)}{a(s)}\, ds\right), \qquad Q(x) := \frac{1}{a(x)W(x)} \int_{x_0}^{x} W(s)\, ds,$$

$$R(x) \;\; := \;\; W(x) \int_{x_0}^{x} \frac{1}{a(s)W(s)}\, ds,$$

where $x \in J$ and x_0 is fixed in J. The boundary point r_2 is said to be

$$\begin{array}{rll} \textit{regular} & \text{if} & Q \in L^1(x_0, r_2), \quad R \in L^1(x_0, r_2); \\ \textit{exit} & \text{if} & Q \notin L^1(x_0, r_2), \quad R \in L^1(x_0, r_2); \\ \textit{entrance} & \text{if} & Q \in L^1(x_0, r_2), \quad R \notin L^1(x_0, r_2); \\ \textit{natural} & \text{if} & Q \notin L^1(x_0, r_2), \quad R \notin L^1(x_0, r_2). \end{array}$$

Analogous definitions can be given for r_1 by considering the interval (r_1, x_0) instead of (x_0, r_2).

Previous classification of the endpoints allows us to state Feller's theorem, which characterizes when the operator A with the so-called Wentzell boundary conditions (i.e. $\lim\limits_{x \to r_1, x \to r_2} Au(x) = 0$) generates a Feller semigroup (see e.g. [5, Chapter VI Theorems 4.14, 4.17]), as follows.

Proposition 3.1. *The operator A with domain*

$$D_M(A) := \{u \in C(\overline{J}) \cap C^2(J) : \; Au \in C(\overline{J})\}$$

generates a Feller semigroup on $C(\overline{J})$ if and only if r_1 and r_2 are of entrance or natural type. The operator A with domain

$$D(A) := \{u \in C(\overline{J}) \cap C^2(J) : \lim_{x \to r_1, x \to r_2} Au(x) = 0\}$$

generates a Feller semigroup on $C(\overline{J})$ if and only if both the endpoints r_1 and r_2 are not of entrance type.

If we are working in the space $C[0, +\infty]$, then the endpoints 0 and $+\infty$ are not of entrance type for the operator $L_{\theta,a}$ defined by

$$L_{\theta,a}\, u(x) \;:=\; x^{2a}\, u''(x) \;+\; (ax^{2a-1} + \theta x^a)\, u'(x),$$

acting on $C[0, +\infty]$ and so the following theorem holds ([7, Theorem 2]).

Theorem 3.2. *For any $\theta \in \mathbb{R}$, $a \in \mathbb{R}$ the operator $L_{\theta,a}$ with domain*

$$D(L_{\theta,a}) = \{u \in C[0, +\infty] \cap C^2(0, +\infty) \; : \lim_{x \to 0^+, x \to +\infty} L_{\theta,a} u(x) = 0\}$$

generates a Feller semigroup on $C[0, +\infty]$.

Similar arguments work as well even if we replace $C(\overline{\mathbb{R}}_+)$ by $C(\overline{\mathbb{R}}_-)$ and the operator $L_{\theta,a}$ by the operator $\widetilde{L}_{\theta,a}\, u = (-x)^{2a}\, u'' + (-a(-x)^{2a-1} + \theta\, (-x)^a)\, u'$ having domain

$$D(\widetilde{L}_{\theta,a}) = \{u \in C(\overline{\mathbb{R}}_-) \cap C^2(-\infty, 0) \; : \lim_{x \to -\infty, x \to 0^-} \widetilde{L}_{\theta,a}\, u(x) = 0\}.$$

Indeed, we prove the following result.

Theorem 3.3. *The operator $\widetilde{L}_{\theta,a}$ with domain*

$$D(\widetilde{L}_{\theta,a}) = \{u \in C(\overline{\mathbb{R}}_-) \cap C^2(-\infty, 0) : \lim_{x \to -\infty, x \to 0^-} \widetilde{L}_{\theta,a}\, u(x) = 0\}$$

generates a positive contraction semigroup in $C(\overline{\mathbb{R}}_-)$.

Proof. In order to study the boundary $-\infty$, let us take $x_0 = -1$. Let us consider the cases **(i)** $a = 1$, **(ii)** $a < 1$, and **(iii)** $a > 1$.
(i) For $x < 0$ and $\theta \in \mathbb{R}$, let us evaluate

$$W_\theta(x) = \exp\left[-\int_{-1}^{x} \frac{t + \theta(-t)}{(-t)^2}\, dt\right] = \exp\left[-\int_{-1}^{x} \frac{\theta - 1}{(-t)}\, dt\right]$$

$$= \exp\left[(\theta - 1)\log(-t)|_{-1}^{x}\right] = (-x)^{\theta-1}.$$

Consequently, for $x < 0$ and $\theta \in \mathbb{R}$ we obtain

$$Q_\theta(x) = \frac{1}{(-x)^{1+\theta}} \int_{-1}^{x} (-t)^{\theta-1}\, dt \quad \text{and} \quad R_\theta(x) = \frac{1}{(-x)^{1-\theta}} \int_{-1}^{x} (-t)^{-\theta-1}\, dt.$$

In particular, for $\theta = 0$ we have

$$Q_0(x) = \frac{\log(-x)}{x} = R_0(x).$$

Therefore $Q_0 \notin L^1(-\infty, -1)$, $R_0 \notin L^1(-\infty, -1)$, and hence $-\infty$ is natural. For $x < 0$ and $\theta \neq 0$ we obtain that

$$Q_\theta(x) = \frac{1 - (-x)^\theta}{\theta(-x)^{1+\theta}} = R_{-\theta}(x).$$

Thus $-\infty$ is natural for any $\theta \in \mathbb{R}$.

(ii) For $x < 0$ let us evaluate

$$W_\theta(x) = \exp\left[-\int_{-1}^{x} \frac{(-a)(-t)^{2a-1} + \theta\,(-t)^a}{(-t)^{2a}}\, dt\right]$$

$$= \exp\left[-\int_{-1}^{x} \left(\frac{-a}{(-t)} + \frac{\theta}{(-t)^a}\right) dt\right]$$

$$= \exp\left[-a\,\log(-t) + \theta\,\frac{(-t)^{1-a}}{1-a}\right]_{-1}^{x}$$

$$= \exp\left[-a\,\log(-x) + \theta\,\frac{(-x)^{1-a}}{1-a} - \frac{\theta}{1-a}\right]$$

$$= \frac{e^{-\frac{\theta}{1-a}\,(1-(-x)^{1-a})}}{(-x)^a}.$$

In particular, for $\theta = 0$, $W_0(x) = \dfrac{1}{(-x)^a}$,

$$Q_0(x) = \frac{(-x)^a}{(-x)^{2a}} \int_{-1}^{x} \frac{1}{(-s)^a}\, ds = \frac{1}{(-x)^a}\left[-\frac{(-s)^{1-a}}{1-a}\right]_{-1}^{x} = \frac{1}{(-x)^a}\left[1 - \frac{(-x)^{1-a}}{1-a}\right],$$

and

$$R_0(x) = \frac{1}{(-x)^a} \int_{-1}^{x} \frac{(-s)^a}{(-s)^{2a}}\, ds = \frac{1}{(-x)^a} \int_{-1}^{x} \frac{1}{(-s)^a}\, ds = Q_0(x).$$

Thus $Q_0 \notin L^1(-\infty,-1)$, $R_0 \notin L^1(-\infty,-1)$, and hence $-\infty$ is natural. For $\theta \neq 0$ we have

$$\begin{aligned}
Q_\theta(x) &= \frac{(-x)^a\, e^{\frac{\theta}{1-a}\,(1-(-x)^{1-a})}}{(-x)^{2a}} \int_{-1}^{x} W_\theta(s)\,ds \\
&= \frac{e^{\frac{\theta}{1-a}\,(1-(-x)^{1-a})}}{(-x)^a} \int_{-1}^{x} \frac{e^{-\frac{\theta}{1-a}\,(1-(-s)^{1-a})}}{(-s)^a}\,ds \\
&= \frac{e^{\frac{\theta}{1-a}\,(1-(-x)^{1-a})}\, e^{-\frac{\theta}{1-a}}}{(-x)^a} \int_{-1}^{x} \frac{e^{\frac{\theta}{1-a}\,(-s)^{1-a}}}{(-s)^a}\,ds \\
&= \frac{e^{\frac{-\theta}{1-a}\,(-x)^{1-a}}}{(-x)^a} \left[-\frac{e^{\frac{\theta}{1-a}\,(-s)^{1-a}}}{\theta}\right]_{-1}^{x} \\
&= -\frac{e^{-\frac{\theta}{1-a}\,(-x)^{1-a}}}{\theta\,(-x)^a} \left[e^{\frac{\theta}{1-a}\,(-x)^{1-a}} - e^{\frac{\theta}{1-a}}\right] \\
&= \frac{1}{\theta\,(-x)^a} \left[e^{\frac{\theta}{1-a}\,(1-(-x)^{1-a})} - 1\right],
\end{aligned}$$

and

$$\begin{aligned}
R_\theta(x) &= \frac{e^{-\frac{\theta}{1-a}\,(1-(-x)^{1-a})}}{(-x)^a} \int_{-1}^{x} \frac{(-s)^a\, e^{\frac{\theta}{1-a}\,(1-(-s)^{1-a})}}{(-s)^{2a}}\,ds \\
&= \frac{e^{-\frac{\theta}{1-a}\,(1-(-x)^{1-a})}}{(-x)^a} \int_{-1}^{x} \frac{e^{\frac{\theta}{1-a}\,(1-(-s)^{1-a})}}{(-s)^a}\,ds \\
&= \frac{e^{\frac{\theta}{1-a}\,(-x)^{1-a}}}{(-x)^a} \int_{-1}^{x} \frac{e^{-\frac{\theta}{1-a}\,(-s)^{1-a}}}{(-s)^a}\,ds = Q_{-\theta}(x).
\end{aligned}$$

Hence $Q_\theta \notin L^1(-\infty,-1)$, $R_\theta \notin L^1(-\infty,-1)$, and $-\infty$ is natural.

(iii) Similar calculations as in the case **(ii)** yield that for any $\theta \in \mathbb{R}$ we have $R_\theta(x) = Q_{-\theta}(x)$, and $Q_\theta \in L^1(-\infty,-1)$, $R_\theta \in L^1(-\infty,-1)$. We conclude that $-\infty$ is regular.

Then, in any case, $-\infty$ and 0 are not of entrance type and the assertion holds. □

Now we focus on the operator $L_{\theta,1}$ (i.e. $a = 1$) acting on the closed subspace $C_0(\mathbb{R}_+)$.
Let us define the mapping $\Phi : \mathbb{R} \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, such that for any $t \in \mathbb{R}$, $x \in \mathbb{R}_+$

$$\Phi(t,x) = \Phi_t(x) = x\,e^t.$$

It is straightforward to show that $\Phi_0(x) = x$, $\Phi_t \circ \Phi_s = \Phi_{t+s}$, $x \in \mathbb{R}_+$, $t,s \in \mathbb{R}$, and, for any $x \in \mathbb{R}_+$, the mapping $t \longmapsto \Phi_t(x)$ is continuous.

According to [11, B-II, Propositions 3.8, 3.13], the operators $S(t)f := f \circ \Phi_t$, $t \in \mathbb{R}$, define positive bounded operators on $C_0(\mathbb{R}_+)$ and $(S(t))_{t\in\mathbb{R}}$ is a positive (C_0) automorphism group on $C_0(\mathbb{R}_+)$. Its generator is the closure of the operator $A_\infty u(x) := xu'(x),\ \ x \in \mathbb{R}_+$ with domain $D(A_\infty) := C_c^1[0,\ +\infty)$. Here

$$C_c[0,+\infty) := \{f \in C(\mathbb{R}_+) : f \text{ vanishes in a neighborhood of } +\infty\},$$

$$C_c^k[0,+\infty) := \{f \in C^k(\mathbb{R}_+) : f \text{ vanishes in a neighborhood of } +\infty\},$$

$$Mf(x) := xf(x), \qquad\qquad x \in \mathbb{R}_+,\ f \in C(\mathbb{R}_+).$$

Observe that the domain of $\overline{A}_\infty$ is

$$D(\overline{A}_\infty) := \left\{u \in C^1(\mathbb{R}_+) \cap C_0(\mathbb{R}_+) : Mu' \in C_0(\mathbb{R}_+)\right\}$$

and $G := \overline{A}_\infty$ generates a (C_0) group of isometries on $C_0(\mathbb{R}_+)$. Indeed for any $t \in \mathbb{R}$,

$$\|S(t)u\|_\infty = \sup_{x\geq 0} |u(xe^t)| = \sup_{s\geq 0} |u(s)| = \|u\|_\infty.$$

Let us consider the square of the operator A_∞, say A_∞^2, given by

$$A_\infty^2\, u(x) = x(xu')'$$

with domain $D(A_\infty^2) := C_c^2[0,+\infty)$ and the square of $\overline{A}_\infty$, say $\overline{A}_\infty^2$, whose domain is

$$D(\overline{A}_\infty^2) := \left\{u \in C^2(\mathbb{R}_+) \cap C_0(\mathbb{R}_+) : Mu',\, M(Mu')' \in C_0(\mathbb{R}_+)\right\}.$$

It is clear that $\overline{A}_\infty^2 = \overline{A_\infty^2}$. In addition, according to Theorem 2.13, $G^2 = \overline{A}_\infty^2$ generates a cosine function and the analytic semigroup $(T(t))$ generated by G^2 has the following Romanov representation:

$$T(t)u(x) = \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-\frac{y^2}{4t}} \left[\frac{S(y)u(x) + S(-y)u(x)}{2}\right] dy$$

$$= \frac{1}{2\sqrt{\pi t}} \int_0^\infty e^{-\frac{y^2}{4t}} \left[u(x\,e^y) + u(x\,e^{-y})\right] dy,$$

for any t with $Re\ (t) > 0$, and $u \in C_0(\mathbb{R}_+),\ \ x \in \mathbb{R}_+$. Then $(T(t))_{t\geq 0}$ is a C_0 semigroup of contractions, since for $t > 0$,

$$|T(t)u(x)| = \left|\frac{1}{2\sqrt{\pi t}} \int_0^\infty e^{-\frac{y^2}{4t}} \left[u(x\,e^y) + u(x\,e^{-y})\right] dy\right|$$

$$\leq \left(\frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-\frac{y^2}{4t}}\, dy\right) \|u\|_\infty = \|u\|_\infty.$$

This implies

$$\|T(t)u\|_\infty = \sup_{x\geq 0} |T(t)u(x)| \leq \|u\|_\infty.$$

Therefore, we have proved the following result

Theorem 3.4. *The closure of the operator* $(A_\infty,\ D(A_\infty))$ *defined by*

$$A_\infty\, u(x) \;=\; xu',$$

with domain $D(A_\infty) := C_c^1[0,\ +\infty)$*, generates a positive* (C_0) *group of isometries on* $C_0(\mathbb{R}_+)$*. Hence the square* $\left(\overline{A}_\infty^{\,2},\ D\left(\overline{A}_\infty^{\,2}\right)\right)$*, with domain*

$$D(\overline{A}_\infty^{\,2}) \;:=\; \left\{u \in C^2(\mathbb{R}_+) \cap C_0(\mathbb{R}_+)\ :\ Mu',\ M(Mu')' \in C_0(\mathbb{R}_+)\right\},$$

generates a cosine function, and an analytic semigroup $(T(t))_{Re(t)>0}$ *of contractions having the following representation:*

$$T(t)u(x) \;=\; \frac{1}{2\sqrt{\pi t}}\int_0^{+\infty} e^{-\frac{y^2}{4t}}\left[u(x\,e^y) \;+\; u(x\,e^{-y})\right]\ dy.$$

In order to consider the operator $L_{\theta,1}$, we shall examine additional properties of the operator $\overline{A}_\infty$ with respect to its square $\overline{A}_\infty^{\,2}$. Let us remark that the operator $\overline{A}_\infty$ is dissipative on $C_0(\mathbb{R}_+)$. This follows from Theorem 2.6 but we give a separate direct proof. Indeed, let $u \in D(\overline{A}_\infty)$ and choose $x_0 \in [0,\infty)$ such that $u(x_0) = e^{i\theta}\|u\|_\infty$, for some real θ.

If $x_0 > 0$, then $u'(x_0) = 0$ and this implies $\overline{A}_\infty\ u(x_0) = 0$. Hence

$$<\overline{A}_\infty\ u,\ \delta_{x_0}> \;=\; 0$$

and since $\delta_{x_0} e^{-i\theta}\ \|u\|_\infty \in j(u)$, we are done.

In the case $x_0 = 0$, $u(0) = \|u\|_\infty$, and u is real valued, then for any $x > 0$ $u(x) \le u(0)$. Thus $\limsup_{x\to 0} u'(x) \le 0$ and so $\overline{A}_\infty\ u(0) \le 0$. Therefore, we can conclude that $\overline{A}_\infty$ is dissipative, in case u is real valued. The proof for u complex is a trivial variant and we omit it.

Notice that

$$\overline{L}_{\theta,1} \;=\; \overline{A_\infty^2 \;+\; (1+\theta)\,A_\infty} \;=\; \overline{A}_\infty^{\,2} \;+\; (1+\theta)\,\overline{A}_\infty.$$

Similar arguments as in [5, Chapter III, Example 2.2] imply that $(1+\theta)\,\overline{A}_\infty$ is $\overline{A}_\infty^{\,2}$ - bounded with $\overline{A}_\infty^{\,2}$ - bound equal to 0. Moreover [5, Chapter III, Lemma 2.4] yields that $\overline{A}_\infty^{\,2} + (1+\theta)\,\overline{A}_\infty$ with domain $D\left(\overline{A}_\infty^{\,2}\right)$ is closed and, as a consequence of [5, Chapter III, Theorem 2.7], generates a contraction semigroup on $C_0(\mathbb{R}_+)$, which is analytic in the right half plane. In addition, the semigroup generated by $\overline{A}_\infty^{\,2} \;+\; (1+\theta)\,\overline{A}_\infty$ is given by

$$U(t) \;=\; T(t)\,S((1+\theta)\,t), \qquad t \ge 0.$$

Hence, if $u \in C_0(\mathbb{R}_+)$ the semigroup has the explicit representation

$$\begin{aligned}
U(t)\,u(x) &= \frac{1}{2\sqrt{\pi t}}\int_0^\infty e^{-\frac{y^2}{4t}}\,[S(y)\,S((1+\theta)t)\,u(x) + S(-y)\,S((1+\theta)\,t)\,u(x)]\,dy\\
&= \frac{1}{2\sqrt{\pi t}}\int_0^\infty e^{-\frac{y^2}{4t}}\,[S(y+(1+\theta)\,t)\,u(x) + S(-y+(1+\theta)\,t)\,u(x)]\,dy\\
&= \frac{1}{2\sqrt{\pi t}}\int_0^\infty e^{-\frac{y^2}{4t}}\,[u\left(x\,e^{y+t(1+\theta)}\right) + u\left(x\,e^{-y+t(1+\theta)}\right)]\,dy.
\end{aligned}$$

This is valid for all $t > 0$ and $x \in \mathbb{R}_+$. Therefore the following result holds.

Theorem 3.5. *The closure of the operator $L_{\theta,1}$ with domain $D\left(\overline{A}_\infty^2\right)$ generates a positive contraction analytic semigroup $(U(t))_{t\in\mathbb{R}_+}$ on $C_0(\mathbb{R}_+)$ having the following explicit representation:*

$$U(t)\,u(x) \;=\; \frac{1}{2\sqrt{\pi t}}\int_0^\infty e^{-\frac{y^2}{4t}}\left[u\left(x\,e^{y+t(1+\theta)}\right)+u\left(x\,e^{-y+t(1+\theta)}\right)\right]dy,$$

for all t with Re $t>0$, and all $x\in\mathbb{R}_+$.

4. Explicit representations in $C(\overline{\mathbb{R}})$

Let us consider the Banach space $C(\overline{\mathbb{R}})$, equipped with the sup-norm, and the operator $L_{\theta,a}$ defined in the Introduction. For $a=0$, the operator $L_{\theta,0}$ is given by

$$L_{\theta,0}u(x)=u''(x)+\theta u'(x).$$

It is well known that the operator $Gu(x) := u'(x)$ with domain

$$D(G):=\{u\in C(\overline{\mathbb{R}}):\; u'\in C(\overline{\mathbb{R}})\}$$

generates the translation group $(S(t))_{t\in\mathbb{R}}$ on $C(\overline{\mathbb{R}})$, where $S(t)u(x) := u(x+t)$, $x,t\in\mathbb{R}$. Hence, according to Theorem 2.13, the square G^2 with domain

$$D(G^2) \;=\; \{u\in C(\overline{\mathbb{R}}) \;:\; u',u''\in C(\overline{\mathbb{R}})\}$$

generates a cosine function and the (analytic) semigroup generated by G^2 has the following representation

$$T(t)u(x) \;=\; \frac{1}{2\sqrt{\pi t}}\int_0^\infty e^{-\frac{y^2}{4t}}\left[u(x+y)+u(x-y)\right]dy$$

for any $t>0$, $u\in C(\overline{\mathbb{R}})$, and $x\in\mathbb{R}$. In addition, according to [6], any $u\in D(G^2)$ satisfies

$$\lim_{x\to\pm\infty}u'(x) \;=\; 0 \;=\; \lim_{x\to\pm\infty}u''(x).$$

It follows that, for any $\theta\in\mathbb{R}$, the semigroup $(U(t))_{t\geq 0}$ generated by $L_{\theta,0} = G^2+\theta G$ in $C(\overline{\mathbb{R}})$ can be written as $T(t)\,S(\theta t)$ and we have

$$U(t)u(x) \;=\; \frac{1}{2\sqrt{\pi t}}\int_0^\infty e^{-\frac{y^2}{4t}}\left[u(x+\theta t+y)+u(x+\theta t-y)\right]dy$$

for any $t>0$, $u\in C(\overline{\mathbb{R}})$ and $x\in\mathbb{R}$.

Our next aim is to show that our operator $L_{\theta,a}$ can be interpreted as an operator of the type $G_a^2+\theta G_a$, where G_a *in some sense* generates a suitable group on $C(\overline{\mathbb{R}})$.

Definition 4.1. Let x be a real number and a be a positive number, then we define

$$x^{\{a\}} \;=\; \begin{cases} x^a & \text{if} \quad x\geq 0;\\ -(-x)^a & \text{if} \quad x<0.\end{cases}$$

Observe that $x^{\{1\}}=x$ for any real number x.

Lemma 4.2. *For any real number x and any $a > 0$ and $b > 0$, we have*

$$\left(x^{\{a\}}\right)^{\{b\}} = x^{\{ab\}},$$

and

$$(-x)^{\{a\}} = -x^{\{a\}}.$$

Moreover, for any $x \neq 0$

$$\frac{d}{dx}\left(x^{\{a\}}\right) = a\,|x|^{a-1}.$$

Proof. Concerning the first assertion, if $x \geq 0$, it is trivial. If $x < 0$, then we have $x^{\{a\}} < 0$, and thus

$$\left(x^{\{a\}}\right)^{\{b\}} = (-(-x)^a)^{\{b\}} = -((-x)^a)^b = -(-x)^{ab} = x^{\{ab\}}.$$

The second and third part of the assertion easily follow from the definition of $x^{\{a\}}$. □

Now we are in position to prove

Theorem 4.3. *Let us assume that $0 < a < 1$ and for any $h \in C(\overline{\mathbb{R}})$, $x \in \mathbb{R}$, define for any $t \in \mathbb{R}$:*

$$T(t)\,h(x) = h\left(\left[x^{\{1-a\}} + (1-a)\,t\right]^{\{\frac{1}{1-a}\}}\right).$$

Then $(T(t))_{t\in\mathbb{R}}$ is a positive contraction group of operators on $C(\overline{\mathbb{R}})$. In addition, for any $h \in C(\overline{\mathbb{R}})$ and $x \in \mathbb{R}$, the mapping $t \to T(t)h(x)$ is continuous, and for any $h \in C(\overline{\mathbb{R}}) \cap C^1(\mathbb{R})$ and $x \in \mathbb{R}$ there exists $\frac{d}{dt}(T(t)h(x))|_{t=0} = |x|^a\, h'(x)$.

Proof. First observe that for any $t \in \mathbb{R}$, $T(t)$ is a linear bounded operator on $C(\overline{\mathbb{R}})$, which is positive and contractive. It is also clear that for any $h \in C(\overline{\mathbb{R}})$ and $x \in \mathbb{R}$ we have

$$T(0)\,h(x) = h\left(\left(x^{\{1-a\}}\right)^{\{\frac{1}{1-a}\}}\right) = h(x)$$

by virtue of Lemma 4.2. This yields that $T(0) = I$.

Let us proceed to show that for any $t, s \in \mathbb{R}$, $h \in C(\overline{\mathbb{R}})$

$$T(t+s)\,h(x) = T(t)\,T(s)\,h(x), \quad x \in \mathbb{R}. \tag{4.1}$$

Indeed,

$$\begin{aligned}
T(t)\,T(s)\,h(x) &= T(t)\,h\left(\left[x^{\{1-a\}} + (1-a)\,s\right]^{\{\frac{1}{1-a}\}}\right) \\
&= h\left(\left(\left(\left[x^{\{1-a\}} + (1-a)\,s\right]^{\{\frac{1}{1-a}\}}\right)^{\{1-a\}} + (1-a)\,t\right)^{\{\frac{1}{1-a}\}}\right) \\
&= h\left(\left[x^{\{1-a\}} + (1-a)\,s\right] + (1-a)\,t\right)^{\{\frac{1}{1-a}\}}
\end{aligned}$$

$$= h\left(\left[x^{\{1-a\}} + (1-a)(t+s)\right]\right)^{\left\{\frac{1}{1-a}\right\}}$$

$$= T(t+s)\,h(x).$$

Thus (4.1) is proved. Now let us observe that for any $h \in C(\overline{\mathbb{R}})$ and $x \in \mathbb{R}$

$$T(t)h(x) - h(x) = h\left(\left[x^{\{1-a\}} + (1-a)\,t\right]^{\left\{\frac{1}{1-a}\right\}}\right) - h(x).$$

It follows that $\lim\limits_{t\to 0} T(t)h(x) - h(x) = 0$. This gives the continuity of the mapping $t \to T(t)h(x)$ at $t = 0$. In addition, there easily follows the continuity at any $t \in \mathbb{R}$. Finally, if $h \in C(\overline{\mathbb{R}}) \cap C^1(\mathbb{R})$, $x \in \mathbb{R}$ and $t \in \mathbb{R}$, $t \neq 0$, let us examine $\dfrac{T(t)h(x) - h(x)}{t}$. We have

$$\frac{T(t)h(x) - h(x)}{t} = \frac{h\left(\left[x^{\{1-a\}} + (1-a)\,t\right]^{\left\{\frac{1}{1-a}\right\}}\right) - h(x)}{t}.$$

Then, taking the limit as $t \to 0$, an easy consequence of de l'Hospital rule gives $\lim\limits_{t\to 0} \dfrac{T(t)h(x) - h(x)}{t} = h'(x)\,|x|^a$ and the proof has been completed. □

Corollary 4.4. *Let us assume that* $0 < a < 1$ *and that* $(T(t))_{t\in\mathbb{R}}$ *is the group of operators on* $C(\overline{\mathbb{R}})$ *defined in Theorem 4.3 and let us denote by*

$$C_e(\overline{\mathbb{R}}) := \{h \in C(\overline{\mathbb{R}}) : h \textit{ is even}\}.$$

Then for any $h \in C_e(\overline{\mathbb{R}})$, *we have*

$$T(t)\,h(-x) = T(-t)\,h(x), \qquad t \in \mathbb{R},\ x \in \mathbb{R}. \tag{4.2}$$

Proof. Let us fix $h \in C(\overline{\mathbb{R}})$, h even (i.e. $h(-x) = h(x)$, for any $x \in \mathbb{R}$). In order to prove (4.2) we observe that

$$T(t)\,h(-x) = h\left(\left[(-x)^{\{1-a\}} + (1-a)\,t\right]^{\left\{\frac{1}{1-a}\right\}}\right)$$

$$= h\left(\left[-x^{\{1-a\}} + (1-a)\,t\right]^{\left\{\frac{1}{1-a}\right\}}\right)$$

$$= h\left(\left[x^{\{1-a\}} - (1-a)\,t\right]^{\left\{\frac{1}{1-a}\right\}}\right)$$

$$= T(-t)\,h(x).$$

□

Corollary 4.5. *For any* $f \in C(\overline{\mathbb{R}}_+)$ *let us denote by* $\widetilde{f}$ *the even extension of* f *to* $\overline{\mathbb{R}}$. *Let* $a \in (0,1)$ *and* $(T(t))_{t\in\mathbb{R}}$ *be the group of operators on* $C(\overline{\mathbb{R}})$ *defined in Theorem 4.3. Then the family of operators* $(\widetilde{T}(t))_{t\in\mathbb{R}}$ *defined as follows*

$$\widetilde{T}(t)\,f(x) = T(t)\,\widetilde{f}(x), \qquad t \in \mathbb{R},\ x \in \mathbb{R}_+,$$

is a positive contraction group on $C(\overline{\mathbb{R}}_+)$*. In addition, for any* $f \in C(\overline{\mathbb{R}}_+)$ *and* $x \in R_+$ *the mapping* $t \to T(t)f(x)$ *is continuous, and for any* $f \in C(\overline{\mathbb{R}}_+) \cap C^1(\mathbb{R}_+)$ *and* $x \in \mathbb{R}_+$ *there exists* $\frac{d}{dt}T(t)f(x)\,|_{t=0} = x^a\, f'(x)$*.*

Proof. From Corollary 4.4, for any $f \in C(\overline{\mathbb{R}}_+)$ we deduce that

$$\widetilde{T}(t)\, f(x) \;= \begin{cases} T(t)\, f(x), & t \geq 0, x \geq 0; \\ T(-t)\, \widetilde{f}(-x), & t < 0, x \geq 0. \end{cases}$$

Hence, by taking into account Theorem 4.3 and the previous Corollary, the assertion holds. □

Note that the previous groups $(T(t))_{t\in\mathbb{R}}$ (respectively, $(\widetilde{T}(t))_{t\in\mathbb{R}}$) on $C(\overline{\mathbb{R}})$ (respectively, on $C(\overline{\mathbb{R}}_+)$), have some regularity properties, as the continuity of the map $t \to T(t)f(x)$ (respectively, $t \to \widetilde{T}(t)f(x)$), for any fixed $f \in C(\overline{\mathbb{R}})$, $x \in \mathbb{R}$ (respectively, $f \in C(\overline{\mathbb{R}}_+)$, $x \in \mathbb{R}_+$) and the pointwise convergence of $\frac{T(h)f(x) - f(x)}{h}$ (respectively $\frac{\widetilde{T}(h)f(x) - f(x)}{h}$), as $h \to 0$, to $Af(x)$ (respectively to $\widetilde{A}f(x)$). Also, $(T(t))_{t\in\mathbb{R}}$ (respectively, $(\widetilde{T}(t))_{t\in\mathbb{R}}$) is strongly measurable on $C(\overline{\mathbb{R}})$ (respectively, on $C(\overline{\mathbb{R}}_+)$), hence strongly continuous at t for all $t \neq 0$ (see [10, Theorem 10.2.3]). Finally, by using the results by Priola [12, Chapter 6], we can conclude that our groups are C_0-groups on the respective spaces, provided that we identify the elements of $C(\overline{\mathbb{R}})$ (respectively $C(\overline{\mathbb{R}}_+)$) with the elements of $C(\mathbb{R}^c)$ (respectively $C(\mathbb{R}^c_+)$). Here $\mathbb{R}^c$ (respectively $\mathbb{R}^c_+$) denotes the Alexandroff compactification of $\mathbb{R}$ (respectively $\mathbb{R}_+$). For the connections with integrated semigroups see also [1].

All these facts allow us to repeat similar arguments as in Section 3 in order to give an explicit representation of the semigroups on $C(\overline{\mathbb{R}})$ (respectively on $C(\overline{\mathbb{R}}_+)$) generated by the operators

$$L_{\theta,a}u(x) = x^{2a}u''(x) + (a|x|^{2a-1} + \theta|x|^a)u'(x) = G_a^2u(x) + (1+\theta)G_au(x), \quad x \in \mathbb{R},$$

where $(G_a, D(G_a))$ is defined as follows

$$D(G_a) = \{u \in C(\overline{\mathbb{R}}) \cap C^1(\mathbb{R}) \,:\, u'(\cdot)|x|^a \in C(\overline{\mathbb{R}})\},$$
$$G_au(x) = u'(x)|x|^a, \quad u \in D(G_a), \quad x \in \mathbb{R}$$

(respectively,

$$\widetilde{L}_{\theta,a}u(x) = x^{2a}u''(x) + (a\, x^{2a-1} + \theta\, x^a)u'(x) = \widetilde{G}_a^2u(x) + (1+\theta)\widetilde{G}_au(x), \quad x \in \mathbb{R}_+,$$

where $(\widetilde{G}_a, D(\widetilde{G}_a))$ is defined as follows

$$D(\widetilde{G}_a) = \{f \in C(\overline{\mathbb{R}}_+) \cap C^1(\mathbb{R}_+) \,:\, f'(\cdot)x^a \in C(\overline{\mathbb{R}}_+)\},$$
$$\widetilde{G}_af(x) = f'(x)\, x^a, \quad f \in D(\widetilde{G}_a), \quad x \in \mathbb{R}_+).$$

More precisely, we have that the operator $L_{\theta,a}$ has domain $D(G_a^2)$, and, respectively, the operator $\widetilde{L}_{\theta,a}$ has domain $D(\widetilde{G_a}^2)$. Consequently, $L_{\theta,a}$ generates the semigroup $(U(t))_{t\geq 0}$ given by

$$U(t)u(x) = \frac{1}{2\sqrt{\pi t}} \int_0^\infty e^{-\frac{y^2}{4t}} \left[T(y)T((1+\theta)t)u(x) + T(-y)T((1+\theta)t)u(x)\right] dy,$$

for all $t > 0$, $u \in C(\overline{\mathbb{R}})$ and $x \in \mathbb{R}$. In a similar way $\widetilde{L}_{\theta,a}$ generates the semigroup $(\widetilde{U}(t))_{t\geq 0}$ given by

$$\widetilde{U}(t)u(x) = \frac{1}{2\sqrt{\pi t}} \int_0^\infty e^{-\frac{y^2}{4t}} \left[\widetilde{T}(y)\widetilde{T}((1+\theta)t)u(x) + \widetilde{T}(-y)\widetilde{T}((1+\theta)t)u(x)\right] dy,$$

for all $t > 0$, $u \in C((\overline{\mathbb{R}_+})$ and $x \in \mathbb{R}_+$.

This gives an explicit representation of $(U(t))_{t\geq 0}$ and $(\widetilde{U}(t))_{t\geq 0}$. Indeed, for any $y \geq 0$, $t > 0$, $u \in C(\overline{\mathbb{R}})$, $x > 0$ such that $x^{1-a} + (1-a)(1+\theta)t \geq (1-a)y$ we define

$$\begin{aligned} I_1(x,y) &:= T(y)\,T((1+\theta)t)u(x) + T(-y)\,T((1+\theta)t)u(x) \\ &= T(y)\,u([x^{1-a} + (1-a)(1+\theta)t]^{\frac{1}{1-a}}) + T(-y)\,u([x^{1-a} + (1-a)(1+\theta)t]^{\frac{1}{1-a}}) \\ &= u([x^{1-a} + (1-a)[(1+\theta)t + y]]^{\frac{1}{1-a}}) + u([x^{1-a} + (1-a)[(1+\theta)t - y]]^{\frac{1}{1-a}}). \end{aligned}$$

Analogously, for $x > 0$ with $x^{1-a} + (1-a)(1+\theta)t < (1-a)y$ we have

$$\begin{aligned} I_2(x,y) &:= T(y)\,T((1+\theta)t)u(x) + T(-y)\,T((1+\theta)t)u(x) \\ &= u([x^{1-a} + (1-a)[(1+\theta)t + y]]^{\frac{1}{1-a}}) + u(-[-x^{1-a} - (1-a)[(1+\theta)t - y]]^{\frac{1}{1-a}}). \end{aligned}$$

Then, for any $t > 0$, $u \in C(\overline{\mathbb{R}})$ and $x > 0$ it yields that

$$U(t)u(x) = \frac{1}{2\sqrt{\pi t}} \int_0^z e^{-\frac{y^2}{4t}} I_1(x,y)\,dy + \int_z^\infty e^{-\frac{y^2}{4t}} I_2(x,y)\,dy,$$

where $z := \dfrac{x^{1-a} + (1-a)(1+\theta)t}{1-a}$. On the other hand, for $x < 0$ with $|x|^{1-a} \leq (1-a)(1+\theta)t$, we have

$$U(t)u(x) = \frac{1}{2\sqrt{\pi t}} \int_0^w e^{-\frac{y^2}{4t}} I_3(x,y)\,dy + \int_w^\infty e^{-\frac{y^2}{4t}} I_4(x,y)\,dy,$$

where $w := \frac{-|x|^{1-a} + (1-a)(1+\theta)t}{1-a}$, and

$$\begin{aligned} &I_3(x,y) := \\ &u([-|x|^{1-a} + (1-a)[(1+\theta)t + y]]^{\frac{1}{1-a}}) + u([-|x|^{1-a} + (1-a)[(1+\theta)t - y]]^{\frac{1}{1-a}}), \\ &I_4(x,y) := \\ &u([-|x|^{1-a} + (1-a)[(1+\theta)t + y]]^{\frac{1}{1-a}}) + u(-[|x|^{1-a} - (1-a)[(1+\theta)t - y]]^{\frac{1}{1-a}}). \end{aligned}$$

In a similar way one can describe explicitly the case $x < 0$ with $|x|^{1-a} > (1-a)(1+\theta)t$. In analogy, by taking into account Corollary 4.5, one can describe $\widetilde{U}(t)u(x)$ for any $t > 0$, $u \in C(\overline{\mathbb{R}}_+)$, $x \geq 0$.

In [9] we apply the results obtained in this paper to some problems arising in financial mathematics.

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