An analogue of oka’s theorem for weakly normal complex spaces

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William Allen Adkins, Aldo Andreotti and John Vincent Leahy
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WILLIAM A. ADKINS, ALDO ANDREOTTI, J. V. LEAHY

Two well known results concerning normal complex spaces are the following. First, the singular set of a normal complex space has codimension at least two. Second, this property characterizes normality for complex spaces which are local complete intersections. This second result is a theorem of Abhyankar [1] which generalizes Oka’s theorem. The purpose of this paper is to prove analogues of these facts for the class of weakly normal complex spaces, which were introduced by Andreotti-Norguet [3] in a study of the space of cycles on an algebraic variety. A weakly normal complex space can have singularities in codimension one, but it will be shown that an obvious class of such singularities is generic.

1. Preliminaries. All complex spaces are assumed to be reduced. If $X$ is a complex space, there is the sheaf $\mathcal{O}_X$ of holomorphic functions on $X$, and the sheaf $\mathcal{O}_X^c$ of $c$-holomorphic functions on $X$. A section of $\mathcal{O}_X^c$ on an open subset $U$ of $X$ is a continuous function $f: U \to \mathbb{C}$ such that $f$ is holomorphic on the regular points of $U$. The complex space $X$ is said to be weakly normal if $\mathcal{O}_X^c = \mathcal{O}_X$. Examples of weakly normal spaces are normal spaces and unions of submanifolds of $\mathbb{C}^m$ in general position.

Let $V_j = \{(x_1, \ldots, x_m) \in \mathbb{C}^m : x_k = 0 \text{ for } n < k < j \text{ and } j < k \leq m\}$ where $n \leq j \leq m$. Then $V_j$ is an $n$-dimensional linear subspace of $\mathbb{C}^m$. Let

$$V_{(n,m)} = \bigcup_{j=n}^m V_j = \{(x_1, \ldots, x_m) \in \mathbb{C}^m : x_ix_j = 0 \text{ for } n \leq i < j \leq m\}$$

and let $S(V_{(n,m)})$ be the singular set of $V_{(n,m)}$.

LEMMA. $V_{(n,m)}$ is a weakly normal complex space and $\dim S(V_{(n,m)}) = n - 1$.

Proof. Since $S(V_{(n,m)}) = \{(x_1, \ldots, x_m) \in \mathbb{C}^m : x_n = \cdots = x_m = 0\}$, $\dim S(V_{(n,m)}) = n - 1$. Let $f: V_{(n,m)} \to \mathbb{C}$ be a continuous function which is holomorphic on the regular points of $V_{(n,m)}$. To prove weak normality of $V_{(n,m)}$, we need to show that $f$ is holomorphic. Let $f_j = f|_{V_j}$. By the Riemann extension theorem, $f_j$ is holomorphic on the $n$-plane $V_j$ and
thus $f_j = f_j(x', x_i)$ is a convergent power series, where $x' = (x_1, \ldots, x_{n-1})$ and $x_i$ are coordinates on $V_j$. Since $f_j|_{x_i=0} = f_k|_{x_a=0}$ for $n \leq j, k \leq m$, we let $f_0(x') = f_j(x', 0)$ and set $g_j(x', x_i) = f_j(x', x_i) - f_0(x')$ for $n \leq j \leq m$. Then $f(x_1, \ldots, x_m) = f_0(x') + \sum_{j=1}^{m} g_j(x', x_i)$ and hence $f$ is holomorphic on $V_{(n,m)}$.

If $X$ is a complex space with $\dim X = n$, let $Sg(X) = S(X) \cup (\cup_{0 \leq k < n} X^{(k)})$ where $S(X)$ is the singular set of $X$ and $X^{(k)}$ is the analytic subset of $X$ defined by $X^{(k)} = \{ x \in X : X$ has a branch of dimension $k$ at $x \}$. If $C_d(X, x)$ denotes the fourth Whitney tangent cone of $X$ at $x$, then Stutz [6] has shown that $W_4 = Sg(X) \cap \{ x \in X : \dim C_d(X, x) > n \}$ is an analytic subset of $X$ of codimension at least two.

2. Codimension one singularities of weakly normal spaces. Let $X$ be a complex space. A point $x \in X$ is said to be an elementary point of type $(n, m)$, for $n \leq m$, if the germ $(X, x)$ is isomorphic to the germ $(V_{(n,m)}, 0)$. Note that if $x \in X$ is an elementary point of type $(n, m)$, then the germ $(X, x)$ is of pure dimension $n$ and the imbedding dimension of $(X, x)$ is $m$. The set of elementary points of $X$ contains the set of regular points of $X$, i.e. the elementary points of type $(n, n)$ for some $n$. In addition, it contains a particularly simple class of singular points of $X$. If $x$ is an elementary point of type $(n, m)$ with $n < m$, then $x$ is singular and $\dim (S(X), x) = n - 1 = \dim (X, x) - 1$.

If $\dim X = n$, let $Y = \cup_{0 \leq k < n} X^{(k)}$ and let $X_1 = \overline{X \setminus Y}$. By a theorem of Remmert, $X_1$ is an analytic set of pure dimension $n$. Let $X_s$ denote the set of all elementary points of $X$ of type $(n, m)$ for some $m$ with $m \geq n = \dim X$. Hence $X_1 \subseteq X_1$ and $X_s$ contains the regular points of $X$ of maximal dimension.

**Theorem 1.** Let $X$ be a weakly normal complex space. Then $A = X_1 \setminus X_s$ is an analytic subset of $X_1$ of codimension at least 2.

**Proof.** Let $n = \dim X$. If $\dim S(X) \leq n - 2$ then $A = X_1 \cap S(X)$. Hence $A$ is analytic and codimension $A \geq 2$. Now suppose that $\dim S(X) = n - 1$. We will show that $A = X_1 \cap (Sg(Sg(X)) \cup W_4)$. Since $Sg(Sg(X)) \cup W_4$ is an analytic set of codimension at least 2 in $X$ and since $\dim X = \dim X_1$, this will prove the theorem.

Let $x \in X_s$. If $x$ is a regular point of $X$, then $x \not\in Sg(Sg(X)) \cup W_4$. If $x$ is an elementary point of type $(n, m)$ where $m > n$, then $\dim C_d(X, x) = n$. Hence $x \not\in W_4$. Moreover, $S(X)$ is a manifold of dimension $n - 1$ in a neighborhood of $x$. Thus $x \not\in Sg(Sg(X))$. Hence $X_s \subseteq X_1 \setminus (Sg(Sg(X)) \cup W_4)$ and $X_1 \cap (Sg(Sg(X)) \cup W_4) \subseteq A$.

Now suppose that $x_0 \in X_1 \cap S(X) \cap (X_1 \setminus (Sg(Sg(X)) \cup W_4))$. Thus
$x_0 \in \text{Sg}(X) \setminus \text{Sg}(\text{Sg}(X))$ and $\dim C_4(X, x_0) = n$. Note also that the germ $(X, x_0)$ is of pure dimension $n$. Since the result to be proved is local, we may assume that $X \subseteq C'$. By Proposition 4.2 of Stutz [6], there is a neighborhood $N$ of $x_0$ in $X$, a polydisc $D \subseteq C^n$, and a choice of coordinates $x_1, \ldots, x_n$ in $C^n$ and $y_1, \ldots, y_t$ in $C^t$ centered at $x_0$ with the following properties.

If $B_0, \ldots, B_r$ are the global branches of $X \cap N$, then for each $j$ ($0 \leq j \leq r$) there is a holomorphic map $f_j: D \to B_j$ such that

(a) $f_j$ is a homeomorphism;
(b) with respect to the coordinates $x_1, \ldots, x_n, y_1, \ldots, y_t$, $f_j(0) = 0$ and

$$f_j(x) = (x_1, \ldots, x_{n-1}, x_n^{p_j}, f_{n+1,j}(x), \ldots, f_t(x))$$

where $p_j$ is a positive integer for $0 \leq j \leq r$;
(c) $f_j(x_1, \ldots, x_n) = \sum_{\nu=0}^{\infty} f_j^{(\nu)}(x_1, \ldots, x_{n-1}) \cdot x_n^\nu$ for $n + 1 \leq i \leq t$ and $0 \leq j \leq r$.

Let $g_j: B_j \to D$ be the continuous inverse of $f_j$ and define a map $h: X \cap N \to C^{n+r}$ by $\pi_i \circ h \mid_{B_j} = g_j$ where $\pi_i: C^{n+r} \to C_{x_1, \ldots, x_{n-1}, x_n}$ is the natural linear projection onto the $n$-plane with coordinates $x_1, \ldots, x_{n-1}, x_{n+p}$ for $0 \leq j \leq r$. To see that the map $h$ is well defined, note first that $S(X)$ is an $n-1$ dimensional manifold in a neighborhood of $x_0$. Furthermore, $B_j \cap B_k \subseteq S(X) \cap N$ for all $j, k$. But $f_j(x', 0) = (x', 0, \ldots, 0) = f_k(x', 0)$ where $x' = (x_1, \ldots, x_{n-1})$. Therefore, if $N$ is chosen small enough, then $B_j \cap B_k = S(X) \cap N = \{y_n = \cdots = y_t = 0\}$ for $0 \leq j, k \leq r$. For each $(y_1, \ldots, y_t) \in S(X) \cap N$, it follows that $g_j(y) = (y_1, \ldots, y_{n-1}, 0)$ for $0 \leq j \leq r$. Thus $h$ is a well defined continuous map.

Since the jacobian matrix $\partial f_j / \partial x$ is given by

$$\frac{\partial f_j}{\partial x} = \begin{bmatrix} I_{n-1} & 0 \\ 0 & p_j x_n^{p_j-1} \\ * & * \end{bmatrix}$$

$h$ is holomorphic on the regular points of $X \cap N$. Since $X$ is weakly normal and $h$ is a homeomorphism onto its image, it follows that $h$ is biholomorphic. Therefore $x_0$ is an elementary singularity of type $(n, n + r)$. Hence $A \subseteq X_1 \cap (\text{Sg}(\text{Sg}(X))) \cup W_4$ and the theorem is proved.

**Remark.** Let $X$ be a weakly normal complex space and suppose that $\text{codim} S(X) = 1$. Theorem 1 shows that there is an elementary singularity of type $(n, m)$ where $m > n$. Since such a singular point is not normal, Theorem 1 implies the well-known theorem that $\text{codim} S(X) \geq 2$ for a normal complex space $X$. 
Theorem 2. Let $X$ be a pure dimensional local complete intersection. Then $X$ is weakly normal if and only if $\text{codim } X \setminus X_i \geq 2$.

Proof. Let $A = X \setminus X_i$. If $X$ is weakly normal then $\text{codim } A \geq 2$ by Theorem 1.

Conversely, suppose $\text{codim } A \geq 2$. Since $X \setminus A = X_i$, the germ $(X, x)$ is weakly normal for each $x \in X \setminus A$. Since $X$ is a pure dimensional local complete intersection, $pf(\mathcal{O}_{X,x}) = \dim X$ for each $x \in X$, where $pf = \text{profondeur}$. From the Hartog theorem for weak normality [2], we conclude that $X$ is weakly normal.

Remarks. (1) For the case of curves, the assumption of local complete intersection is not needed. A curve $X$ is weakly normal if and only if $X \setminus X_i = \emptyset$. An algebraic proof of this fact was given by Bombieri [5].

(2) If $X$ is a pure dimensional hypersurface in $\mathbb{C}^{n+1}$, then Theorem 2 can be proved without the use of the Hartog theorem for weak normality. This case follows from the result of Becker in [4].

(3) Let $X \subseteq \mathbb{C}^{n+1}$ be a pure dimensional hypersurface. If $X$ is weakly normal, there is another characterization of $X \setminus X_i$ than that which is given by the proof of Theorem 1. This description is as follows. There is a holomorphic function $f \in \mathcal{O}(\mathbb{C}^{n+1})$ such that $X = V(f) = \{x \in \mathbb{C}^{n+1}: f(x) = 0\}$ and such that there is a sheaf equality $(f) \cdot \mathcal{O} = \mathcal{I}_X$ where $\mathcal{I}_X$ is the sheaf of ideals of $X$. Then

$$S(X) = \left\{ x \in X : \frac{\partial f}{\partial z_i}(x) = 0 \text{ for } 1 \leq i \leq n+1 \right\}.$$ 

At a point $x_0 \in S(X)$ the Hessian form is defined by

$$H(f)_{x_0}(u) = \sum_{i,j=1}^{n+1} \frac{\partial^2 f}{\partial z_i \partial z_j}(x_0) \cdot u_i u_j.$$ 

Let $\mu(x_0) = \text{rank } H(f)_{x_0}$ and set $S_2(X) = \{ x \in S(X) : \mu(x) \leq 1 \}$.

Claim. If $X$ is weakly normal and $\dim S(X) = n - 1$, then

$$W_4 \cap (S(X) \setminus \text{Sg}(S(X))) = S_2(X) \setminus S(X) \setminus \text{Sg}(S(X)).$$

Proof. From the proof of Theorem 1, $X \setminus X_i = \text{Sg}(S(X)) \cup W_4$. Suppose $x \in S(X) \setminus \text{Sg}(S(X))$ but $x \not\in W_4$. Then the proof of Theorem 1 shows that $x$ is an elementary singular point of type $(n, n+1)$. A proper choice of local coordinates about $x$ shows that $(X, x)$ is isomorphic to $(V(z_1 z_2), 0)$. Hence $\mu(x) = 2$ and $x \not\in S_2(X)$. 


Now suppose that \( x \in S(X) \setminus S_g(S(X)) \) but \( x \not\in S_g(S(X)) \). Thus \( \mu(x) \geq 2 \). If \( \mu(x) > 2 \) then the implicit function theorem shows that \( \dim(S(X), x) \leq n - 2 \). Therefore \( \mu(x) = 2 \) and choosing convenient local coordinates centered at \( x \) gives \( f(z) = az_1z_2 + O(3) \) where \( a \neq 0 \). Hence \( x \) is an elementary singular point of type \((n, n + 1)\). Therefore, \( x \not\in W_4 \) and the claim is proved.

For weakly normal hypersurfaces this claim gives an easy differential criterion for computing the portion of the set \( W_4 \) which is contained in \( S(X) \setminus S_g(S(X)) \). This claim is false for hypersurfaces which are not weakly normal.

**Example.** Let \( X = \{(x, y, z) \in \mathbf{C}^3: x^2 - zy^2 = 0\} \) be the Cayley umbrella in \( \mathbf{C}^3 \). Then \( X \setminus X_s = \{(0, 0, 0)\} \) so that \( X \) is weakly normal by Theorem 2. Remark (3) then shows that \( W_4 = \{0, 0, 0\} \).

**References**


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