Journal of Stochastic Analysis

Volume 1 | Number 2 Article 5

June 2020

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Journal of Stochastic Analysis
Vol. 1, No. 2 (2020) Article 5 (10 pages)
DOI: 10.31390/josa.1.2.05

VON NEUMANN'S MINIMAX THEOREM FOR CONTINUOUS QUANTUM GAMES

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Abstract. The concept of a classical player, corresponding to a classical random variable, is extended to include quantum random variables in the form of self adjoint operators on infinite dimensional Hilbert space. A quantum version of Von Neumann's Minimax theorem for infinite dimensional (or continuous) games is proved.

1. Introduction: Classical Two-person Zero-sum Games

In classical *zero-sum infinite-dimensional (or continuous) games* between two players, called Blue and Red, each player has an infinite number of *moves* (or *pure strategies*) available in each play of the game. The moves of Blue and Red are identified [5] with the points of some closed and bounded intervals $[a, b] \subset \mathbb{R}$ and $[c, d] \subset \mathbb{R}$ respectively. To the case when Blue makes choice $\lambda \in [a, b]$ and Red makes choice $l \in [c, d]$ we assign a numerical non-negative *payoff* $Z(\lambda, l)$ to Blue and a corresponding payoff $-Z(\lambda, l)$ to Red. We assume that both players take a conservative approach, in the sense that Blue wants to maximize his minimum payoff and Red wants to minimize the maximum payoff to Blue. If there exists $(\lambda_0, l_0) \in [a, b] \times [c, d]$ such that

$$\max_{\lambda} \min_{l} Z(\lambda, l) = Z(\lambda_0, l_0) = \min_{l} \max_{\lambda} Z(\lambda, l)$$

then (λ_0, l_0) is a *saddle point* of Z and λ_0, l_0 are *optimal moves* for Blue and Red respectively. If Z has no saddle point then Blue and Red must use *mixed strategies* i.e, see [5], they must alter their moves and choose them using random devises represented by classical probability distribution functions

$$F : [a, b] \to [0, 1]\,, \; G : [c, d] \to [0, 1]$$

associated with the probability measure spaces $(\Omega_1, \sigma_1, \mu_1)$ and $(\Omega_2, \sigma_2, \mu_2)$, respectively, that describe the outcomes of the random devices used, respectively, by Blue and Red. Here, for $i = 1, 2$, the Ω_i's are the sample spaces, the μ_i's are the probability measures on the Ω_i 's , and the σ_i's are the corresponding σ-algebras

Received 2020-5-18; Accepted 2020-6-14; Communicated by Hui-Hsiung Kuo.
2010 *Mathematics Subject Classification.* Primary 46N10, 46N30, 81P16, 91A05 ; Secondary 81Q10, 91A40.
Key words and phrases. Quantum game, self-adjoint operator, quantum random variable, trace class operator, density operator, quantum state, Kakutani's fixed point theorem, minimax theorem.
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of measurable subsets of the Ω_i's. We may then think of the *classical players* Blue and Red as random variables B and R, i.e., as measurable functions

$$B : (\Omega_1, \sigma_1, \mu_1) \to [a, b]\,,\; R : (\Omega_2, \sigma_2, \mu_2) \to [c, d]\ . \tag{1.1}$$

Then

$$F(\lambda) = \mu_1 \left(\{\omega_1 \in \Omega_1 : B(\omega_1) \leq \lambda\}\right) = \Pr\left(\text{ player } B \text{ makes a move } \leq \lambda\right) \tag{1.2}$$

and

$$G(l) = \mu_2 \left(\{\omega_2 \in \Omega_2 : R(\omega_2) \leq l\}\right) = \Pr\left(\text{ player } R \text{ makes a move } \leq l\right)\ . \tag{1.3}$$

If we let the double Riemann-Stieltjes integral

$$K(F, G) = \int_c^d \int_a^b Z(\lambda, l)\, dF(\lambda)\, dG(l)$$

be the *total expected payoff to B*, then the fundamental problem of two-person zero-sum continuous game theory is the existence of probability distributions, i.e., of mixed strategies, F^* and G^* such that

$$\max_F \min_G K(F, G) = K(F^*, G^*) = \min_G \max_F K(F, G)\ . \tag{1.4}$$

If such F^* and G^* exist, then $K(F^*, G^*)$ is the *value* of the game. If the payoff function Z is continuous then, by the extension of von Neumann's Minimax Theorem [14] to infinite dimensional games [5, 2], such F^* and G^* always exist. For a historical study of von Neumann's Minimax Theorem we refer to [12].

Modern proofs of the existence of optimal strategies typically use Kakutani's fixed point theorem [11] for strategies in $\mathbb{R}^n$ or, in the case of infinite dimensional sets of strategies, its extensions to Banach or locally convex topological vector spaces [8, 13]. In all cases, a compact and convex set of strategies from which to choose, is required.

The remaining sections are structured as follows:

In Section 2 we describe how the classical concept of a *player* can be translated into the language of quantum mechanics [10, 6, 9, 15], i.e., in terms of self-adjoint operators on Hilbert space, also referred to as *observables* or *quantum random variables*, due to the fact that their spectrum is real. In Section 3 we describe Kakutani's fixed point theorem for Banach spaces, in the form that we are going to use it [11, 13, 8, 16]. In Section 4 we describe a class of compact and convex sets of positive operators of trace one, that will serve as our sets of mixed quantum strategies available to the two players. In Section 5 we describe how the classical two-person zero-sum game setup can be formulated in terms of: self-adjoint operators on Hilbert space, the spectral theorem, and positive operators of unit trace. We then prove the quantum version of the Minimax Theorem, i.e., Theorem 5.1. We remark that finite dimensional two-person zero-sum quantum games were considered in [3].

2. From Classical to Quantum Players

For a probability measure space (Ω, σ, μ) as in Section 1, $L^2(\Omega, \sigma, \mu)$, denotes the Hilbert space of all μ- equivalence classes of square-integrable complex-valued functions f defined on Ω with inner product

$$\langle f, g \rangle = \int_\Omega \bar{f}(\omega)\, g(\omega)\, d\mu(\omega) \ .$$

To pass from the classical to a quantum formulation of game theory we notice that with the classical random variables, i.e., with the classical players B and R of (1.1), we can associate self-adjoint multiplication operators

$$\mathcal{B} : L^2(\Omega_1, \sigma_1, \mu_1) \to L^2(\Omega_1, \sigma_1, \mu_1) \ , \ \mathcal{R} : L^2(\Omega_2, \sigma_2, \mu_2) \to L^2(\Omega_2, \sigma_2, \mu_2)$$

defined pointwise by

$$\mathcal{B}(f)(\omega_1) = B(\omega_1)\, f(\omega_1) \ , \ \mathcal{R}(g)(\omega_2) = R(\omega_2)\, g(\omega_2) \ .$$

In general, for $f \in L^2(\Omega, \sigma, \mu)$ we let $\rho = |f\rangle\langle f|$ denote the operator

$$\rho : L^2(\Omega, \sigma, \mu) \to L^2(\Omega, \sigma, \mu)$$

defined by

$$\rho(g) = |f\rangle\langle f|(g) = \langle f, g\rangle f$$

and for $\lambda \in \mathbb{R}$ we let $E(\lambda)$ denote the projection operator

$$E(\lambda) : L^2(\Omega, \sigma, \mu) \to L^2(\Omega, \sigma, \mu)$$

defined by

$$E(\lambda)(g) = \chi_{(-\infty, \lambda]}\, g \ .$$

If $f \equiv 1$ then $\rho = |f\rangle\langle f|$ is a *state* i.e a positive operator of unit trace and, in analogy to (1.2),

$$F(\lambda) := \operatorname{tr} \rho E(\lambda) = \langle 1, E(\lambda) 1\rangle = \mu\left((-\infty, \lambda]\right) = \Pr\left(\text{player} B \text{ makes a move} \leq \lambda\right) \ .$$

It is then suggested that we think of a *quantum player* as a self-adjoint operator

$$T = \int_{\mathbb{R}} \lambda \ dE(\lambda)$$

on some infinite dimensional separable Hilbert space $\mathcal{H}$, whose available moves coincide with its spectrum $\sigma(T)$, with the projection $E(\lambda)$ interpreted as the event *player* T *makes a move* $\leq \lambda$, and with probability distribution

$$F : \lambda \in \mathbb{R} \to F(\lambda) = \operatorname{tr} \rho E(\lambda) \in [0, 1] \ ,$$

determined by a state ρ on $\mathcal{H}$.

We denote by $m(T)$ and $M(T)$ the *lower bound* and *upper bound* of T, respectively, defined by

$$m(T) = \inf_{\|x\|=1} \langle Tx, x\rangle \ , \ M(T) = \sup_{\|x\|=1} \langle Tx, x\rangle \ .$$

If T is bounded then $m(T)$ and $M(T)$ are finite and the spectrum $\sigma(T)$ is contained in the interval $I_T = [m(T), M(T)]$. In particular $m(T), M(T) \in \sigma(T)$. We recall that $\sigma(T)$ is closed in $\mathbb{R}$.

3. Kakutani's Fixed Point Theorem

Following [16], if $\mathcal{S}$ is a subset of a normed space $\mathcal{V}$ then a set-valued mapping $U : \mathcal{S} \to P(\mathcal{S})$, where $P(\mathcal{S})$ is the *power set* of $\mathcal{S}$, is a *K-mapping of $\mathcal{S}$ into itself* if:

(i) for each x in $\mathcal{S}$, $U(x)$ is a compact convex non-empty subset of $\mathcal{S}$;
(ii) the *graph* of U, $\mathcal{G}(U) = \{(x, y) : y \in U(x)\}$ is closed in $\mathcal{S} \times \mathcal{S}$.

Condition (ii) is equivalent to the following *upper semi-continuity* condition:

(iii) if $x_n \to x$ in $\mathcal{S}$, $y_n \in U(x_n)$ and $y_n \to y$ then $y \in U(x)$.

A *fixed point* of a *K-mapping* U is a point $x \in \mathcal{S}$ such that $x \in U(x)$. A subset $\mathcal{S}$ of a normed space $\mathcal{V}$ has *the Kakutani property* if each K-mapping U of $\mathcal{S}$ into $\mathcal{S}$ has a fixed point. Kakutani's fixed point theorem [11] states that every compact convex nonempty subset of $\mathbb{R}^n$ has the Kakutani property. The theorem was extended in [1] from $\mathbb{R}^n$ to any Banach space $\mathcal{V}$.

4. A Compact Set of Quantum States

For an infinite dimensional separable Hilbert space $\mathcal{H}$ we denote by $\mathcal{B}(\mathcal{H})$ the Banach algebra of bounded linear operators $A : \mathcal{H} \to \mathcal{H}$, with the usual operator norm $\|A\|$, and by $\mathcal{T}(\mathcal{H})$ the Banach space of *trace class* operators $T : \mathcal{H} \to \mathcal{H}$ with the *trace norm*

$$\|T\|_1 = \mathrm{Tr}\,|T| = \mathrm{Tr}\,(\sqrt{T^*T}) = \sum_{i=1}^{\infty} \langle e_i, \sqrt{T^*T} e_i \rangle \ ,$$

where $(e_i)_{i=1}^{\infty}$ is any orthonormal basis of $\mathcal{H}$. For $T \in \mathcal{T}(\mathcal{H})$ and $A \in \mathcal{B}(\mathcal{H})$,

$$|\mathrm{Tr}(TA)| \le \|T\|_1 \|A\| \ .$$

We denote by $\mathcal{S}(\mathcal{H})$ the closed convex subset of $\mathcal{T}(\mathcal{H})$ consisting of all *density operators* (or *quantum states*) in $\mathcal{H}$, i.e., all positive operators $\rho : \mathcal{H} \to \mathcal{H}$ with $\mathrm{Tr}(\rho) = 1$. Equipped with the metric

$$d(\rho_1, \rho_2) = \|\rho_1 - \rho_2\|_1$$

the *state space* $\mathcal{S}(\mathcal{H})$ is a complete separable metric space and is a Banach space under $\|\cdot\|_1$. In particular, $\mathcal{S}(\mathcal{H}) \times \mathcal{S}(\mathcal{H})$ is a Banach space under $\|(\rho_1, \rho_2)\| = \|\rho_1\|_1 + \|\rho_2\|_1$. Unlike the finite dimensional case, $\mathcal{S}(\mathcal{H})$ is not compact.

As shown in [10], if $(e_i)_{i=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$, $(c_i)_{i=1}^{\infty} \subseteq \mathbb{R}$ a sequence bounded from below, and

$$\mathcal{D} = \{\psi \in \mathcal{H} : \sum_{i=1}^{\infty} |c_i|^2 \langle e_i, \psi \rangle^2 < \infty\} \ ,$$

then $\mathcal{D}$ is dense in $\mathcal{H}$ and the formula

$$\mathcal{E}(\psi) = \sum_{i=1}^{\infty} c_i \langle e_i, \psi \rangle e_i \tag{4.1}$$

defines a self-adjoint operator $\mathcal{E}$ in $\mathcal{H}$ with domain $\mathcal{D}$. In particular, $\mathcal{E}$ has the e_i's as eigenvectors corresponding to its eigenvalues c_i, $i = 1, 2, \dots$. If the multiplicities of the c_i's are finite and $c_i \to \infty$ as $i \to \infty$ then, by Lemma 11.55 of [10], for an arbitrary positive constant c the set

$$\mathcal{A}(c) = \{\rho \in \mathcal{S}(\mathcal{H}) : \operatorname{Tr} \rho\mathcal{E} \leq c\} \tag{4.2}$$

is a compact subset of the metric space $\mathcal{S}(\mathcal{H})$. By the linearity of the trace, $\mathcal{A}(c)$ is also convex. Typically, $\mathcal{E}$ is the *energy operator* of a quantum oscillator system and we think of $\mathcal{A}(c)$ as the set of *quantum states with mean energy* $\leq c$.

5. The Quantum Minimax Theorem

Based on our concept of a quantum player described in Section 2, we may set up a quantum game as follows:

In the notation of Section 2, for $i = 1, 2$ let $\mathcal{H}_i$ be an infinite dimensional separable Hilbert space, and let

$$\mathcal{B} : \mathcal{H}_1 \to \mathcal{H}_1 \,,\; \mathcal{R} : \mathcal{H}_2 \to \mathcal{H}_2$$

be bounded self-adjoint operators with spectral resolutions

$$\mathcal{B} = \int_{\mathbb{R}} \lambda \; dE(\lambda) \,,\; \mathcal{R} = \int_{\mathbb{R}} l \; dE'(l)$$

respectively. Let

$$Z : I_{\mathcal{B}} \times I_{\mathcal{R}} \to [0, +\infty)$$

be continuous and not identically equal to zero on $I_{\mathcal{B}} \times I_{\mathcal{R}}$. We could assume, although we do not that here, that Z is equal to zero outside $\sigma(\mathcal{B}) \times \sigma(\mathcal{R})$, indicating the impossibility of assigning a profit to non-observable moves.

For quantum states ρ, ϕ on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, for $\lambda, l \in \mathbb{R}$ we define

$$F_\rho(\lambda) = \operatorname{tr} \rho E(\lambda) \,,\; G_\phi(l) = \operatorname{tr} \phi E'(l) \;.$$

With $\mathcal{B}$ and $\mathcal{R}$ we associate the total expected payoff function

$$K(\rho, \phi) = \int_{I_{\mathcal{R}}} \int_{I_{\mathcal{B}}} Z(\lambda, l) \, dF_\rho(\lambda) \, dG_\phi(l) \geq 0 \;.$$

Finally, for given positive constants c_1, c_2, as in (4.2), we denote

$$\mathcal{A}_1(c_1) = \{\rho \in \mathcal{S}(\mathcal{H}_1) : \operatorname{Tr} \rho\mathcal{E}_1 \leq c_1\} \,,\; \mathcal{A}_2(c_2) = \{\phi \in \mathcal{S}(\mathcal{H}_2) : \operatorname{Tr} \phi\mathcal{E}_2 \leq c_2\}$$

where $\mathcal{E}_1, \mathcal{E}_1$ are of the type (4.1). We may now formulate and prove the following quantum version of the Minimax Theorem which provides the quantum analogue of (1.4).

Theorem 5.1. *There exist quantum states* ρ^*, ϕ^* *such that*

$$\max_{\rho \in \mathcal{A}_1(c_1)} \; \min_{\phi \in \mathcal{A}_2(c_2)} \; K(\rho, \phi) = K(\rho^*, \phi^*) = \min_{\phi \in \mathcal{A}_2(c_2)} \; \max_{\rho \in \mathcal{A}_1(c_1)} \; K(\rho, \phi).$$

Proof. For each pair of quantum states ρ and ϕ, the set $I_{\mathcal{B}} \times I_{\mathcal{R}}$ has finite measure

$$(F_\rho(M(\mathcal{B})) - F_\rho(m(\mathcal{B})) \, (G_\phi(M(\mathcal{R})) - G_\phi(m(\mathcal{R}))$$

and Z is continuous (thus measurable) and nonnegative on it. Thus, by Fubini's theorem

$$\int_{I_{\mathcal{R}}} \int_{I_{\mathcal{B}}} Z(\lambda, l) \, dF_\rho(\lambda) \, dG_\phi(l) = \int_{I_{\mathcal{B}}} \int_{I_{\mathcal{R}}} Z(\lambda, l) \, dG_\phi(l) \, dF_\rho(\lambda) \;.$$

For each $\rho \in \mathcal{A}_1(c_1)$, the mapping

$$p_1 : \phi \in \mathcal{A}_2(c_2) \to p_1(\phi) = \int_{I_\mathcal{B}} \int_{I_\mathcal{R}} Z(\lambda, l)\, dG_\phi(l)\, dF_\rho(\lambda) \in \mathbb{R}$$

is continuous. To see that we notice that if (ϕ_n) is a sequence in $\mathcal{A}_2(c_2)$ with $\|\phi_n - \phi\|_1 \to 0$ as $n \to \infty$, where $\phi \in \mathcal{A}_2(c_2)$, then

$$\begin{aligned} |p_1(\phi_n) - p_1(\phi)| =& |\int_{I_\mathcal{B}} \int_{I_\mathcal{R}} Z(\lambda, l)\, d\left(G_{\phi_n} - G_\phi\right)(l)\, dF_\rho(\lambda)| \\ \leq& V\left(G_{\phi_n} - G_\phi\right) V\left(F_\rho\right)\, Z(\lambda_0, l_0) \end{aligned}$$

where

$$Z(\lambda_0, l_0) = \max_{\lambda \in I_\mathcal{B}, l \in I_\mathcal{R}} Z(\lambda, l)$$

and

$$\begin{aligned} V\left(G_{\phi_n} - G_\phi\right) =& \inf\{K : \sum_{i=0}^{k} |\left(G_{\phi_n} - G_\phi\right)(x_i) - \left(G_{\phi_n} - G_\phi\right)(x_{i-1})| \leq K\} \\ V\left(F_\rho\right) =& \inf\{K : \sum_{i=0}^{N} |F_\rho(y_i) - F_\rho(y_{i-1})| \leq K\}\ , \end{aligned}$$

where the inequality must hold for all partitions

$$\{x_0 < x_1 < ... < x_{i-1} < x_i < ... < x_k\}, k \in \mathbb{N},$$

of $I_\mathcal{R}$ and

$$\{y_0 < y_1 < ... < y_{i-1} < y_i < ... < y_N\}, N \in \mathbb{N},$$

of $I_\mathcal{B}$, are the *total variations* (see [7] and [4]) of $G_{\phi_n} - G_\phi$ and F_ρ over $I_\mathcal{R}$ and $I_\mathcal{B}$ respectively. We have

$$\begin{aligned} \sum_{i=0}^{k} |\left(G_{\phi_n} - G_\phi\right)(x_i) - \left(G_{\phi_n} - G_\phi\right)(x_{i-1})| =& \sum_{i=0}^{k} |\mathrm{tr}\,(\phi_n - \phi)(E(x_i) - E(x_{i-1}))| \\ \leq& \sum_{i=0}^{k} \|\phi_n - \phi\|_1 \|E(x_i) - E(x_{i-1})\| \\ =& \|\phi_n - \phi\|_1 \sum_{i=0}^{k} \|E(x_i) - E(x_{i-1})\|. \end{aligned}$$

Since, for each i, $E(x_i) - E(x_{i-1})$ is a self-adjoint operator,

$$\|E(x_i) - E(x_{i-1})\| = \sup_{\|h\|=1} |\langle h, (E(x_i) - E(x_{i-1}))h\rangle|$$

so, by Proposition 1, p. 309 of [17],

$$\begin{aligned} \sum_{i=0}^{k} \|E(x_i) - E(x_{i-1})\| =& \sum_{i=0}^{k} \sup_{\|h\|=1} |\langle h, (E(x_i) - E(x_{i-1}))h\rangle| \\ =& \sup_{\|h\|=1} \sum_{i=0}^{k} |\langle h, (E(x_i) - E(x_{i-1}))h\rangle| \leq 1\ . \end{aligned}$$

Thus

$$V\left(G_{\phi_n} - G_\phi\right) \leq \|\phi_n - \phi\|_1 \ .$$

Similarly

$$V\left(F_\rho\right) \leq \|\rho\|_1 = 1 \ .$$

Thus

$$|p_1(\phi_n) - p_1(\phi)| \leq \|\phi_n - \phi\|_1 \ Z(\lambda_0, l_0)$$

so p_1 is continuous for each $\rho \in \mathcal{A}_1(c_1)$. Thus, since $\mathcal{A}_2(c_2)$ is compact, for each $\rho \in \mathcal{A}_1(c_1)$ there exists a (not necessarily unique) quantum state $\phi^*(\rho) \in \mathcal{A}_2(c_2)$ at which p_1 attains its minimum, i.e., for each $\rho \in \mathcal{A}_1(c_1)$,

$$K(\rho, \phi^*(\rho)) = \min_{\phi \in \mathcal{A}_2(c_2)} K(\rho, \phi) \ , \tag{5.1}$$

Similarly, for each $\phi \in \mathcal{A}_2(c_2)$ there exists a (not necessarily unique) quantum state $\rho^*(\phi) \in \mathcal{A}_1(c_1)$ at which the mapping

$$p_2 : \rho \in \mathcal{A}_1(c_1) \to p_2(\rho) = \int_{I_\mathcal{B}} \int_{I_\mathcal{R}} Z(\lambda, l) \, dG_\phi(l) \, dF_\rho(\lambda) \in \mathbb{R} \ ,$$

attains its maximum, i.e., for each $\phi \in \mathcal{A}_2(c_2)$,

$$K(\rho^*(\phi), \phi) = \max_{\rho \in \mathcal{A}_1(c_1)} K(\rho, \phi) \ . \tag{5.2}$$

On the compact and convex subset $\mathcal{A}_1(c_1) \times \mathcal{A}_2(c_2)$ of $\mathcal{S}(\mathcal{H}) \times \mathcal{S}(\mathcal{H})$, we define the point-to-set mapping U by

$$U(\rho, \phi) = \{(\rho^*(\phi), \phi^*(\rho)) : \text{ such that (5.1) and (5.2) are true}\} \ .$$

We will show that U is a K-mapping:

As shown above, $U(\rho, \phi)$ is non-empty. Let $((\rho_n^*(\phi), \phi_n^*(\rho)))$ be a sequence in $U(\rho, \phi)$ converging to an element (x, y) of $\mathcal{A}_1(c_1) \times \mathcal{A}_2(c_2)$ with respect to the product metric $d \times d$ on $\mathcal{S}(\mathcal{H}) \times \mathcal{S}(\mathcal{H})$. Then, as $n \to +\infty$,

$$\|\rho_n^*(\phi) - x\|_1 \to 0 \, , \ \|\phi_n^*(\rho) - y\|_1 \to 0 \ .$$

We will show that $(x, y) \in U(\rho, \phi)$, so $U(\rho, \phi)$ is a closed (thus compact) subset of $\mathcal{A}_1(c_1) \times \mathcal{A}_2(c_2)$. Equivalently, we will show that x and y have properties (5.2) and (5.1) respectively.

Let $\epsilon > 0$ be given, and let $n_0 \in \mathbb{N}$ be such that, for all $n \geq n_0$,

$$\|\phi_n^*(\rho) - y\|_1 < \frac{\epsilon}{Z(\lambda_0, l_0)} \ , \ \|\rho_n^*(\phi) - x\|_1 < \frac{\epsilon}{Z(\lambda_0, l_0)}$$

Then, as in the proof of continuity of p_1,

$$\begin{aligned} K(\rho, y) \leq & |K(\rho, y) - K(\rho, \phi_{n_0}^*(\rho))| + K(\rho, \phi_{n_0}^*(\rho)) \\ \leq & Z(\lambda_0, l_0) \|\phi_n^*(\rho) - y\|_1 + K(\rho, \phi_{n_0}^*(\rho)) \\ < & \epsilon + K(\rho, f) \end{aligned}$$

for all $f \in \mathcal{A}_2(c_2)$, since $\phi_{n_0}^*(\rho)$ has property (5.1). By the arbitrariness of ϵ, it follows that

$$K(\rho, y) \leq K(\rho, f)$$

for all $f \in \mathcal{A}_2(c_2)$, so y has property (5.1). Similarly,

$$\begin{aligned} K(x,\phi) =& |K(x,\phi) - K(\rho^*_{n_0}(\phi),\phi) + K(\rho^*_{n_0}(\phi),\phi)| \\ \geq & K(\rho^*_{n_0}(\phi),\phi) - |K(x,\phi) - K(\rho^*_{n_0}(\phi),\phi)| \\ \geq & K(\rho^*_{n_0}(\phi),\phi) - Z(\lambda_0,l_0)\|\rho^*_{n_0}(\phi) - x\|_1 \\ > & K(r,\phi) - \epsilon \end{aligned}$$

for all $r \in \mathcal{A}_1(c_1)$, since $\rho^*_{n_0}(\phi)$ has property (5.2). By the arbitrariness of ϵ, it follows that

$$K(x,\phi) \geq K(r,\phi)$$

for all $r \in \mathcal{A}_1(c_1)$, so x has property (5.2). Thus $U(\rho,\phi)$ is compact.

To show that $U(\rho,\phi)$ is also convex, let $(\rho^*_i(\phi),\phi^*_i(\rho)) \in U(\rho,\phi)$, $i = 1,2$, and let $t \in [0,1]$. Then

$$t\,(\rho^*_1(\phi),\phi^*_1(\rho)) + (1-t)(\rho^*_2(\phi),\phi^*_2(\rho)) = (t\,\rho^*_1(\phi) + (1-t)\rho^*_2(\phi), t\,\phi^*_1(\rho) + (1-t)\phi^*_2(\rho)) .$$

By the linearity of the trace,

$$\begin{aligned} K(t\,\rho^*_1(\phi) + (1-t)\rho^*_2(\phi),\phi) =& t\,K(\rho^*_1(\phi),\phi) + (1-t)K(\rho^*_2(\phi),\phi) \\ \geq & t\,K(\rho,\phi) + (1-t)K(\rho,\phi) \\ =& K(\rho,\phi) \end{aligned}$$

for all $\rho \in \mathcal{A}_1(c_1)$, so $t\,\rho^*_1(\phi) + (1-t)\rho^*_2(\phi)$ has property (5.2). Similarly

$$K(\rho, t\,\phi^*_1(\rho) + (1-t)\phi^*_2(\rho)) \leq K(\rho,\phi)$$

for all $\phi \in \mathcal{A}_2(c_2)$, so $t\,\phi^*_1(\rho) + (1-t)\phi^*_2(\rho)$ has property (5.1). Thus $U(\rho,\phi)$ is convex.

To show that U satisfies the upper semi-continuity condition of Section 3, let $((\rho_n,\phi_n))$ be a sequence in $\mathcal{A}_1(c_1) \times \mathcal{A}_2(c_2)$ converging to an element (ρ,ϕ) of $\mathcal{A}_1(c_1) \times \mathcal{A}_2(c_2)$ with respect to the product metric $d \times d$ on $\mathcal{S}(\mathcal{H}) \times \mathcal{S}(\mathcal{H})$, i.e. $\|\rho_n - \rho\|_1 \to 0$ and $\|\phi_n - \phi\|_1 \to 0$ as $n \to +\infty$, and let $(\rho^*_n(\phi_n),\phi^*_n(\rho_n)) \in U(\rho_n,\phi_n)$, i.e. for all $(r,f) \in \mathcal{A}_1(c_1) \times \mathcal{A}_2(c_2)$ we have

$$K(\rho_n,\phi^*_n(\rho_n)) \leq K(\rho_n,f)\,,\; K(\rho^*_n(\phi_n),\phi_n) \geq K(r,\phi_n)\;,$$

with $(\rho^*_n(\phi_n),\phi^*_n(\rho_n))$ converging to an element (ρ^*,ϕ^*) of $\mathcal{A}_1(c_1) \times \mathcal{A}_2(c_2)$ with respect to the product metric $d \times d$ on $\mathcal{S}(\mathcal{H}) \times \mathcal{S}(\mathcal{H})$, i.e. $\|\rho^*_n(\phi_n) - \rho^*\|_1 \to 0$ and $\|\phi^*_n(\rho_n) - \phi^*\|_1 \to 0$ as $n \to +\infty$. We will show that $(\rho^*,\phi^*) \in U(\rho,\phi)$, i.e. that $\rho^* = \rho^*(\phi)$ and $\phi^* = \phi^*(\rho)$, meaning that

$$K(\rho,\phi^*) \leq K(\rho,f)\,,\; K(\rho^*,\phi) \geq K(r,\phi)\;,$$

for all $(r,f) \in \mathcal{A}_1(c_1) \times \mathcal{A}_2(c_2)$.

So let $\epsilon > 0$ be given, and let $n_0 \in \mathbb{N}$ be such that, for all $n \geq n_0$,

$$\|\phi^*_n(\rho_n) - \phi^*\|_1 < \frac{\epsilon}{2\,Z(\lambda_0,l_0)}\,,\; \|\rho_n - \rho\|_1 < \frac{\epsilon}{2\,Z(\lambda_0,l_0)}$$

and

$$\|\rho^*_n(\phi_n) - \rho^*\|_1 < \frac{\epsilon}{2\,Z(\lambda_0,l_0)}\,,\; \|\phi_n - \phi\|_1 < \frac{\epsilon}{2\,Z(\lambda_0,l_0)}\;.$$

Then

$$\begin{aligned} K(\rho,\phi^*) \leq & |K(\rho,\phi^*) - K(\rho_{n_0},\phi^*_{n_0}(\rho_{n_0}))| + K(\rho_{n_0},\phi^*_{n_0}(\rho_{n_0})) \\ < & \frac{\epsilon}{2} + K(\rho_{n_0}, f) \\ \leq & \frac{\epsilon}{2} + |K(\rho_{n_0}, f) - K(\rho, f)| + K(\rho, f) \\ < & \frac{\epsilon}{2} + \frac{\epsilon}{2} + K(\rho, f) \\ = & \epsilon + K(\rho, f) \ , \end{aligned}$$

so $K(\rho,\phi^*) \leq K(\rho, f)$ for all $f \in \mathcal{A}_2(c_2)$. Similarly,

$$\begin{aligned} K(\rho^*,\phi) \geq & K(\rho^*_{n_0}(\phi_{n_0}),\phi_{n_0}) - |K(\rho^*,\phi) - K(\rho^*_{n_0}(\phi_{n_0}),\phi_{n_0})| \\ > & K(r,\phi_{n_0}) - \frac{\epsilon}{2} \\ \geq & K(r,\phi) - |K(r,\phi_{n_0}) - K(r,\phi)| - \frac{\epsilon}{2} \\ > & K(r,\phi) - \frac{\epsilon}{2} - \frac{\epsilon}{2} \\ = & K(r,\phi) - \epsilon \ , \end{aligned}$$

so $K(\rho^*,\phi) \geq K(r,\phi)$ for all $r \in \mathcal{A}_1(c_1)$. Thus U is upper semi-continuous and by Kakutani's theorem U has a fixed point $(\rho^*,\phi^*) \in U(\rho^*,\phi^*)$, i.e. such that

$$K(\rho,\phi^*) \leq K(\rho,\phi) \leq K(\rho^*,\phi) \ ,$$

for all $(\rho,\phi) \in \mathcal{A}_1(c_1) \times \mathcal{A}_2(c_2)$, which is equivalent to

$$\max_{\rho\in\mathcal{A}_1(c_1)} \min_{\phi\in\mathcal{A}_2(c_2)} K(\rho,\phi) = K(\rho^*,\phi^*) = \min_{\phi\in\mathcal{A}_2(c_2)} \max_{\rho\in\mathcal{A}_1(c_1)} K(\rho,\phi).$$

□

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