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SCS 13: Complements to Relations with the Interpolation Properties and Continuous Lattices

Klaus Keimel

Technische Universität Darmstadt, Germany, keimel@mathematik.tu-darmstadt.de

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TOPIC	Complements to relations with the interpolation properties and continuous lattices				
REFERENCE	(1) My MEMO SCS 8-1-76, (2) SCS Memo of LAWSON 7-12-76				

8. Solution of Problem (7.2) in (1).

K.H. Hofmann has pointed out to me, that an affirmative answer to this problem can be drawn from Proposition 4 in (2). We restate the following definition from (2):

3.1) In a complete lattice S we write $x \lll y$, if the following holds:
 For every updirected subset D of S such that $y \leq \sup D$, there is an element d in D such that $x \ll d$.

We note:

- (8.2) If $x \ll y \ll z$, then $x \lll z$.
- (8.3) If $x \lll y$, then $x \ll y$.
- (8.4) $x \lll^* y$ iff $x \ll^* y$ (For \ll^* see sec.5 in 1).

Proposition 4 in (2) asserts:

(8.5) In every compact semilattice, $x \lll y$ iff $x \prec y$, where $x \prec y$ is defined to mean that y is interior to the principal filter generated by x as in sec. 7.

Thus, for compact semilattices we conclude:

- (8.6) If $x \ll y \ll z$, then $x \prec z$.
- (8.7) $x \ll^* y$ iff $x \prec^* y$.

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The last statement shows that the left reflector $\mathcal{C}_\perp \circ K : \underline{CS} \rightarrow \underline{CL}$ constructed in sec. 7 is nothing but the restriction of the left reflector $\mathcal{B}_\perp \circ W' : \underline{L} \rightarrow \underline{CL}$ in sec.6 to the 'subcategory' \underline{CS} of \underline{L} . For the same reason the kernel operators v_S and w_S of (6.5) and (7.11) coincide on a compact semilattice S .

9. Further comments.

Lawson's proposition (8.5) which even would deserve the name LEMMA has another interesting consequence (the equivalence of (1) and (4) generalises (1.25) of ATLAS where both S and T are supposed to be continuous):

(9.1) PROPOSITION. Let S be a compact and T a continuous lattice. Let $g: S \rightarrow T$ preserve arbitrary infima, with right adjoint $d: S \rightarrow T$. Then the following are equivalent:

- (1) g preserves updirected suprema.
- (2) $x \ll_T y \Rightarrow d(x) \ll_S d(y)$.
- (3) $x \lesssim_T y \Rightarrow d(x) \lesssim_S d(y)$.
- (4) g is continuous (topologically).

Proof. (1) \Leftrightarrow (2) by (6.3) and (3) \Leftrightarrow (4) by ATLAS (1.19). Clearly, (4) \Rightarrow (1).

(2) \Rightarrow (3) $x \ll_T y \Leftrightarrow x \ll_T y$ as T is a continuous lattice,
 $\Rightarrow \exists a \in T \ x \ll_T a \ll_T y$ as the interpolation prop. holds in a continuous lattice.

$\Rightarrow d(x) \ll_S d(a) \ll_S d(y)$ by (2)

$\Rightarrow d(x) \lesssim_S d(y)$ by (8.6)

(9.2) Let (L, \sqsubseteq) be a CSRIP-object, and $\kappa: L \rightarrow \mathcal{P}_\perp L$

$$a \mapsto \kappa(a) = \{x \in L \mid x \sqsubseteq a\}.$$

For κ to preserve updirected suprema it is necessary and sufficient that $x \sqsubseteq y \Rightarrow x \ll_L y$. In this case $\kappa: L \rightarrow \mathcal{P}_\perp L$ is surjective. Its right adjoint $\mathbb{I} \mapsto \sup \mathbb{I} : \mathcal{P}_\perp L \rightarrow L$ is injective and the kernel operator

$$u_\perp: a \mapsto \sup \kappa(a) : L \rightarrow L$$

preserves updirected suprema.

Proof. Suppose that c preserves updirected suprema and let $x \sqsubseteq y$. In order to prove $x \ll y$ let D be an updirected subset of L with $y \sqsubseteq \sup D$. Then $x \in c(y) \subseteq c(\sup D) = \bigcup \{c(d) ; d \in D\}$, whence $x \in c(d)$, i.e. $x \sqsubseteq d$ for some $d \in D$. Conversely, suppose that $x \sqsubseteq y \implies x \ll y$. Let D be an updirected subset of L and $a = \sup D$. Clearly, $c(a)$ contains $c(d)$ for all d in D . At the other hand, if $x \in c(a)$, then $x \sqsubseteq a$, whence $x \sqsubseteq a' \sqsubseteq a$ for some a' . Then $a' \ll a = \sup D$ by hypothesis. Consequently, $a' \sqsubseteq d$ for some d in D . We conclude that $x \in c(d)$.

For the surjectivity of c , note that for every ~~finite~~ ideal $I \in \mathcal{I}_\perp(L)$ we have $I = \bigcup \{c(d) ; d \in I\} = c(\sup I)$, if we assume that c preserves updirected suprema. The remaining assertions are then clear. \square

Now let (L, \sqsubseteq) be a SRIP-object (see sec. 3). The map $c : L \rightarrow \mathcal{P}_\perp(L)$ defined as above preserves finite suprema if and only if the following axiom is satisfied:

(9.3) If $x \sqsubseteq a \vee b$, then there are $x_1 \sqsubseteq a, x_2 \sqsubseteq b$ such that $x \sqsubseteq x_1 \vee x_2$.

By (3.17) we conclude:

(9.4) $c : L \rightarrow \mathcal{P}_\perp(L)$ is a SRIP-morphism iff (9.3) holds.

Together with (9.2) this yields:

(9.5) Let (L, \sqsubseteq) be a CSRIP-object. Then $c : L \rightarrow \mathcal{P}_\perp(L)$ is a CSRIP-morphism if and only if (L, \sqsubseteq) satisfies (9.3) and the property " $x \sqsubseteq y \implies x \ll y$ ".

(9.6) Under the conditions of (9.5), $c : L \rightarrow \mathcal{P}_\perp(L)$ not only has a ~~left~~ right adjoint $I \mapsto \sup I : \mathcal{P}_\perp(L) \rightarrow L$, but also a left adjoint $g : \mathcal{P}_\perp(L) \rightarrow L$ given by

$$\begin{aligned}
 g(I) &= \sup \{a \mid c(a) \subseteq I\} && (I \in \mathcal{P}_\perp(L)) \\
 &= \sup \{a \mid x \sqsubseteq a \implies x \in I\} \\
 &= \sup \{a \mid I = c(a)\}
 \end{aligned}$$