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SCS 10: Points with Small Semilattices

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SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

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TOPIC Points with Small Semilattices

REFERENCE SCS Memo of Hofmann and Mislove, 6-28-76.

(1) First of all I would like to call attention to a preprint I have just submitted for publication entitled "Spaces which force a basis of subsemilattices." In this paper it is shown that a topological semilattice has small semilattices at a point p if p has a compact, finite-dimensional, "well-fitted" neighborhood, where "well-fitted" is a technical term describing the behavior of components in a neighborhood of a point. It is defined below. Points in locally connected, totally disconnected, and locally connected X totally disconnected spaces have well-fitted neighborhoods. In fact a rather far-reaching class of finite-dimensional spaces are included.

It is convenient for our purposes to introduce a component operator. Let X be a topological space, $A \subseteq X$, and $p \in A$. Then $C_p(A)$ denotes the component (i.e., maximal connected set) of p in the subspace A.

Definition. Let S be a topological space. If $A \subseteq B \subseteq S$, then A is said to be fitted within B if for each $p \in A$, $C_p(A) = C_p(B) \cap A$.

A neighborhood W of a point $p \in A$ is a fitted neighborhood of p if W is compact and p has a basis of compact neighborhoods, each of which is fitted within W.

A neighborhood W of a point $p \in S$ is a <u>well-fitted</u> neighborhood of p if (i) for each q in the interior of W, W is a fitted neighborhood of q, and (ii) for any $A \subset W$, if $C_p(W) \cap (\bigcup_{a\in A} C_a(W))^* \neq \emptyset$, then $p \in (\bigcup_{a\in A} C_a(W))^*$.

(2) Let me at this point throw in a couple of conjectures. First a definition. The space $\overline{X}$ is said to have <u>local component convergence</u> (l.c.c.) <u>at</u> p if for any neighborhood W of p, there exist neighborhoods V and U of p such that

(1) $V \subset U \subset W$,

(2) If $Q \subset V$ and $C_p(U) \cap (\bigcup_{q\in Q} C_q(W)) \neq \emptyset$, then $p \in (\bigcup_{q\in Q} C_q(W))^*$. Roughly speaking, we are requiring that if components approach the component of p locally, then they actually approach p.

<u>Conjecture 1</u>. Let $S \in$ <u>CS</u>. If $p \in S$, S is l.c.c. at p, and p has a finite-dimensional neighborhood in which components are locally connected, then S has small semilattices at p.

<u>Conjecture 2</u>. Let $S \in$ <u>CS</u>, S finite-dimensional, and suppose the peripheral points in S are closed. Then $S \in$ <u>CL</u>.

Proofs or counter-examples are not easily forthcoming on such problems if past experience is any guide.

(3) Let $S \in \underline{CS}$. Let $\Lambda(S) \subset S$ be all elements of S at which S has small semilattices.

Proposition 1. $\Lambda(S)$ is a sup- subsemilattice of S containing 0 closed under arbitrary sups. Hence in its own order, $\Lambda(S)$ is a complete lattice.

Proof. Let $x, y \in \Lambda(S)$. Then $x = \sup\{a: x \in (\uparrow a)^\circ\}$ and $y = \sup\{b: y \in (\uparrow b)^\circ\}$, and both of these are up-directed sets. Hence $x \vee y = \sup\{a \vee b: x \in (\uparrow a)^\circ$ and $y \in (\uparrow b)^\circ\}$ and $x \vee y \in (\uparrow a)^\circ \cap (\uparrow b)^\circ = (\uparrow a \vee b)^\circ$. Thus $x \vee y \in \Lambda(S)$.

Now suppose x_α is an up-directed net in $\Lambda(S)$ and $x = \sup x_\alpha$. If U is open, $x \in U$, $\exists\, x_\beta \in U$. Since $x_\beta \in \Lambda(S)$, $\exists\, y \in U$ such that $x_\beta \in (\uparrow y)^\circ$. Hence $x \in (\uparrow y)^\circ$. □

Note that this proposition applies nicely to some of the considerations of H and M, Memo 6-28-76, e.g. Proposition 11.

Question: Is $\Lambda(S) \in \underline{CL}$?

(4) Definition. Let A be a topological semilattice, $x \in S$. $\{U_n: n=1,2,\ldots\ldots\}$ is a fundamental system for x if

(1) Each U_n is open;

(2) $U_n \cdot U_n \subset U_{n-1}$, $\overline{U}_n \subset U_{n-1}$

(3) $x \in U_n$ for each n.

<u>Proposition 2</u>. (1) If $\{U_n\}_{n=1}^{\infty}$ is a fundamental system for x, $\bigcap_{n=1}^{\infty} U_n$ is a closed semilattice containing x.

(2) Each neighborhood of x contains a fundamental system for x.

<u>Proposition 3</u>. If $S \in$ <u>CS</u>, then for $x \in S$ and each fundamental system $\lambda = \{U_n\}_{n=1}^{\infty}$, let $x_\lambda = \inf(\bigcap_{n=1}^{\infty} U_n)$. Then if the fundamental systems are ordered by inclusion, x_λ is a net converging upwards to x.

<u>Definition</u>. $y \lll x$ if whenever $\bigvee A \geq x$, there exists $F^{\text{finite}} \subset A \ni y \ll \bigvee F$.

<u>Proposition 4</u>. Let $S \in CS$. Then $y \lll x \Leftrightarrow x \in (\uparrow y)^\circ$.

<u>Proof</u>. $\Leftarrow$ Straightforward [illegible].

$\Rightarrow$ By Prop. 3 $x = \sup x_\lambda$ where λ is a fundamental system. Hence $\exists \lambda = \{U_n\}_{n=1}^{\infty} \ni y \ll x_\lambda$.

For each U_i in λ, pick $x_i \in U_i \setminus \uparrow y$ (we can do this if $x \notin (\uparrow y)^\circ$). Now

$$\begin{aligned} x_i x_{i+1} \cdots x_{i+j} &\in U_i U_{i+1} \cdots U_{i+j} \\ &\subseteq U_i U_{i+1} \cdots (U_{i+j-1})^2 \\ &\vdots \\ &\subset U_i^2 \subset U_{i-1} \end{aligned}$$

Hence $w_i = \bigwedge_{j \geq i} x_j \in \overline{U}_{i-1} \subset U_{i-2}$.

Now w_i is an increasing sequence which must converge up to some w. Since $w_i \in U_{i-2}$, $w \in \bigcap_{i=1}^{\infty} U_i$. Thus $w \geq x_\lambda$.

Since $y \ll x_\lambda$, $\exists\, w_j \ni y \leq w_j$.
But $w_j \leq x_j$ and $y \not\leq x_j$, a contradiction.

So $x \in (\uparrow y)^\circ$. □

Corollary 5. If $x \ll y \ll z$, then $x \lll z$. Hence $w \in \Lambda(S)$ if $w = \sup\{x : x \ll w\}$.