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MEMORY-MODULATED CIR PROCESS WITH DISCRETE DELAY COEFFICIENTS

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ABSTRACT. This paper proposes a stochastic model with discrete delay coefficients for modeling population dynamics. This model is a memory-modulated version of the well known CIR process. Existence and uniqueness of the global solution and non-negativity of the solution are discussed. Significantly, results concerning persistence of the population and other longterm qualitative behaviors of these non-Markov processes are obtained using hitting times. Under particular parameter values, the model exhibits reflection at the zero boundary and so models populations that persist due to the influence of past population.

1. Introduction

Natural processes are often affected by random influences. For example, population dynamics are influenced by environmental noise. Stochastic differential equations provide useful models in these cases. Additionally, stochastic delay differential equations, where the dynamics depend on both current and past states, allow for more complex models. For example, delay models in population biology allow birth and death rates that depend on both current and past population sizes. One example arises in the modeling of a gestation or maturation period for a species so that the growth in the current population of adults depends on the population of adults reproducing at some prior time.

In this paper we consider two stochastic delay models. The first is the (memory-modulated) Squared Bessel Process with Discrete Delay [23],

$$dX_t = \delta X_{t-\tau}\, dt + 2\sqrt{X_t X_{t-\tau}}\, dW_t; \quad X_t = \theta_t \in [0,\infty) \text{ for } t \in [-\tau, 0], \tag{1.1}$$

In this model, the Squared Bessel Process,

$$dX_t = \delta\, dt + 2\sqrt{|X_t|}\, dW_t, \qquad X_0 = x_0, \tag{1.2}$$

is modulated by its past to produce a stochastic delay model. The drift and diffusion coefficients are weighted by the history of the process at a fixed delay interval, τ. The Squared Bessel Process has a unique strong solution and, if δ, $x_0 \geq 0$, then $X_t \geq 0$ [22, p. 439].

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When $\tau = 0$, the modulated process (1.1) reduces to the well known Geometric Brownian Motion which is positive given a positive initial value, justifying the removal of the absolute value in the square root of the square.

The second model we consider is the proposed Memory-Modulated CIR Process with Discrete Delay,

$$dX_t = \gamma X_{t-\tau}\left(K - X_t\right)dt + \alpha\sqrt{X_t X_{t-\tau}}\,dW_t; \quad X_t = \theta_t \in [0,\infty) \text{ for } t \in [-\tau, 0]. \tag{1.3}$$

This model is a memory-modulated version of Feller's Square Root Process,

$$dX_t = \gamma\left(K - X_t\right)dt + \alpha\sqrt{X_t}\,dW_t, \quad X_0 = x_0. \tag{1.4}$$

Feller's Square Root Process has a unique strong solution. If $x_0 > 0$ and $2\gamma K > \alpha^2$, then the solution is positive. This SDE is used in the CIR model of short term interest rates [5] and is also known as the Generalized Squared Bessel Process [11], [22]. Purpose of this modulation is to allow the process to incorporate the past memory which has an effect on the persistence of the process near the zero boundary.

When $\tau = 0$, the modulated process (1.3) reduces to the Stochastic Food Web Model,

$$dX_t = \gamma X_t\left(K - X_t\right)dt + \alpha X_t\,dW_t \quad ; \quad X_0 = x_0. \tag{1.5}$$

When $x_0 > 0$, this SDE has the unique strong explicit solution

$$X_t = \frac{\exp\left\{\left(\gamma K - \alpha^2/2\right)t + \alpha W_t\right\}}{x_0^{-1} + \gamma\int_0^t \exp\left\{\left(\gamma K - \alpha^2/2\right)s + \alpha W_s\right\}ds}$$

and the process never hits zero [9, p. 116]. This model is a stochastic version of the logistic growth model

$$\frac{dX_t}{dt} = \gamma X_t\left(K - X_t\right), \tag{1.6}$$

which was introduced by Pierre-François Verhulst in 1838 and derived again by Alfred J. Lotka. (See Lotka's book *Elements of Physical Biology*, reprinted by Dover in 1956 as *Elements of Mathematical Biology* [15].) This model was later developed further as the Lotka-Volterra model to model predator-prey systems and multi-species environments. William Feller constructed other diffusion models that include the environmental noise in the model [6], [7], [8].

Another model that has been vastly analyzed is the Population Model for Optimal Harvesting from a Stochastic Crowded Environment [16] which is derived from the logistic model (1.6) by randomizing the growth parameter, replacing γ with $\gamma + \alpha\dot{W}_t$, to give the SDE,

$$dX_t = \gamma X_t\left(K - X_t\right)dt + \alpha X_t\left(K - X_t\right)dW_t; \quad X_0 > 0. \tag{1.7}$$

It is possible to prove the existence of a strong solution, uniqueness of the solution, boundedness of the solution (most importantly $X_t \geq 0$) and boundary behavior. If $\alpha^2 K < 2\gamma$, then $\lim_{t\to\infty} X_t = K$ almost surely for all $X_0 > 0$ [16].

Some other stochastic non-delay population models are discussed in [2], [3], [9], [16], [19]. Later, the Lotka-Volterra model was developed into a stochastic delay model [18], [1] by Mao et al. and there are many other models that use the delay to include the effects of the population size from the past.

In this paper we show properties of the (memory-modulated) Squared Bessel Process with Discrete Delay (1.1) and apply similar methods to prove some properties of the proposed Memory-Modulated CIR Process with Discrete Delay (1.3). It is important to note that the proposed models are produced from memory-modulation of well established models in the literature and result in stochastic delay versions of the exponential and logistic population models. We further show that the behavior of the solutions of these models is significantly different from those of Geometric Brownian Motion and the non-delay stochastic versions of the logistic model, (1.5), and (1.7). In particular, our delay models have critical parameter values below which the zero boundary can be reached. In that case, the process reflects at zero and the population persists due to the delay components in the drift terms of the stochastic delay equation.

2. Preliminaries

In this section we discuss some preliminary results used in this paper and some known models from the literature. In the rest of this paper, W_t is a standard Brownian motion, $\mathcal{F}_t$ is the corresponding filtration of σ-algebras, $K > 0$ is the carrying capacity of the environment, $\tau \geq 0$ is the fixed delay time, and α, δ, and γ are positive constants.

The main differences between a deterministic ordinary differential equation and a stochastic delay differential equation are the noise term and the dependence on the history of the process. It is reasonable to use a stochastic process to represent the noise term, so a real valued stochastic delay differential equation (SDDE) driven by standard Brownian motion can be given as,

$$dX_t = a(t, X_{t-\tau}, X_t)\, dt + b(t, X_{t-\tau}, X_t)\, dW_t \tag{2.1}$$

where τ is a fixed delay time, and $a, b : [-\tau, \infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with the initial segment $X_t = \theta_t$ for $t \in [-\tau, 0]$. To deal with the stochastic differentials we use the Itô calculus. See [21, p. 22] for a detailed construction. We will often be using stopping times and the following theorem for stopping times.

Definition 2.1 (Stopping Time [22, p. 42])**.** A *stopping time* T with respect to the filtration $\mathcal{F}_t$ is a map on Ω with values in $[0, \infty]$ such that for every t,

$$\{T \leq t\} \in \mathcal{F}_t.$$

Theorem 2.2 (Optional Stopping Theorem [22, p. 69])**.** *If X is a martingale and S, T are two bounded stopping times with $S \leq T$,*

$$X_S = \mathbb{E}\left[X_T | \mathcal{F}_S\right] \quad a.s.$$

If X is uniformly integrable, the family $\{X_S\}$ where S runs through the set of all stopping times is uniformly integrable and if $S \leq T$,

$$X_S = \mathbb{E}\left[X_T | \mathcal{F}_S\right] = \mathbb{E}\left[X_\infty | \mathcal{F}_S\right] \quad a.s.$$

2.1. Existence and uniqueness of the global solution of a stochastic differential equation. The diffusion coefficients in the equations (1.1) and (1.3) fail to satisfy the Lipschitz condition. They do satisfy the Yamada-Watanabe condition from which uniqueness of solutions can be established [22, p. 390]. Note that in the following, we only impose the Lipschitz or Yamada-Watanabe conditions on the third variable of the coefficient functions. That is because it is possible to consider one lag time $[n\tau, (n+1)\,\tau]$ at a time. The solution to the process in the previous lag time interval $X_{t-\tau}$ is known and hence uniquely determined. It is then possible to construct a global solution [17], [20].

Theorem 2.3. *Suppose the stochastic delay differential equation*

$$dX_t = a(t, X_{t-\tau}, X_t)\,dt + b(t, X_{t-\tau}, X_t)\,dW_t$$

satisfies the following conditions:

(1) *The drift coefficient a satisfies a Lipschitz condition; i.e., for each compact H there exists a constant C_n such that $|a(t,y,x) - a(t,y,z)| \le C_n|x-z|$, for all x, y, and $z \in H$ and $t \in [n\tau, (n+1)\tau]$.*
(2) *The diffusion coefficient b satisfies a Yamada-Watanabe condition; i.e., for each compact H there exist a strictly increasing function $h_n\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\int_0^{0+} h_n^{-2}(x)\,dx = +\infty$ such that, $|b(t,y,x) - b(t,y,z)| \le h_n(|x-z|)$ for all x, y, and $z \in H$ and $t \in [n\tau, (n+1)\tau]$.*

Then pathwise uniqueness holds for the solution of this SDDE.

We also use the following comparison theorem to prove the non-negativeness of our models.

Theorem 2.4 (Comparison theorem [22, p. 394])**.** *Let $a^i, i = 1,2$ be two bounded Borel functions s.t. $a^1 \ge a^2$ everywhere and one of them at least satisfies a Lipschitz condition. If $X^i, i = 1,2$ are solutions to the SDE*

$$dX_t = a^i(t, X_t)\,dt + \sigma(t, X_t)\,dW_t; \quad X_0 = x_0 \tag{2.2}$$

defined on the same space with respect to the same BM and if $X_0^1 \ge X_0^2$ a.s., then

$$P\left[X_t^1 \ge X_t^2 \text{ for all } t \ge 0\right] = 1.$$

2.2. Local time theorem for semimartingales and the occupation time formula. We use following theorems to prove the instantaneous reflection of Squared Bessel Process with Delay. For proof of these theorems we refer to [22].

Theorem 2.5 (Local time theorem for semimartingales [22, p. 225])**.** *For any continuous semimartingale X, there exists a modification of the local time process $\{L_t^a; a \in \mathbb{R}, t \in \mathbb{R}_+\}$ s.t the map $(a,t) \to L_t^a$ is a.s. continuous in t and cadlag in a. Moreover, if X has the decomposition $X = M + V$, then*

$$L_t^a - L_t^{a-} = 2\int_0^t 1_{\{X_s=a\}} dX_s = 2\int_0^t 1_{\{X_s=a\}} dV_s.$$

Theorem 2.6 (Occupation time formula [22, p. 224])**.** *If X is a continuous semi-martingale, there is a P-negligible set outside of which*

$$\int_0^t \Pi\left(X_s\right) d\langle X, X\rangle_s = \int_{-\infty}^{\infty} \Pi\left(a\right) L_t^a \, da$$

for every t and every positive Borel function Π.

3. Proposed Stochastic Delay Model and Properties

In this section we construct our model with drift and diffusion functions that depend not only on current population size but also on the population size at a prior time, a fixed lag time τ earlier. In the discussion of existence and uniqueness of the solution we use the fact that the process up to the previous lag time is known. We further analyze the boundedness, extinction time and asymptotic behavior of the solution. We introduce a new way to analyze the boundary behavior of these non-Markov stochastic discrete delay processes.

3.1. (Memory-modulated) squared Bessel process with discrete delay. We introduce a model below which we call the (memory-modulated) Squared Bessel Process with Discrete Delay (3.1) due to its relation to the Squared Bessel Process (1.2). For comparison purposes we note the following properties of the Squared Bessel Process [22, p. 442 ff.]. For positive integer values of the parameter δ, the Squared Bessel Process is the square of the modulus of a δ-dimensional Brownian motion. The construction of the Squared Bessel Process for general parameter δ and other properties are discussed in [22]. The behavior of this process at the zero boundary depends on the parameter δ:

(1) If $\delta = 0$, zero is an absorbing boundary.
(2) If $\delta > 0$, the process is instantaneously reflecting at zero.
(3) If $0 < \delta < 2$, the process is recurrent and reaches zero a.s.
(4) If $\delta = 2$, the process is recurrent and zero is polar.
(5) If $\delta > 2$ the process is transient and zero is polar.

The non-Markov Squared Bessel Process with Discrete Delay has similar properties. First we show that this process with positive integer parameter δ can be constructed as a squared process [23] and thus is non-negative. Let B_t be δ-dimensional Brownian motion, $\mathcal{F}_t$ be the natural filtration and $\theta_t > 0$ a.e. Let

$$X = Y_1^2 + Y_2^2 + \cdots + Y_\delta^2,$$

where

$$Y_i = \int_0^t \sqrt{X_{s-\tau}} \, dB_s^i.$$

Then using the Itô formula we get

$$dX_t = \delta X_{t-\tau} \, dt + 2\sqrt{X_{t-\tau} X_t} \, dW_t, \quad X_t = \theta_t \text{ for } t \in [-\tau, 0],$$

with one-dimensional Brownian motion, W_t.

We can also define the process for any $\delta \geq 0$ as the solution of the SDDE

$$dX_t = \delta X_{t-\tau} \, dt + 2\sqrt{X_{t-\tau} X_t} \, dW_t; \quad X_t = \theta_t \text{ for } t \in [-\tau, 0]. \tag{3.1}$$

Theorem 3.1. *Let $\tau > 0$ be fixed and let θ_t be positive, integrable, and independent of $\mathcal{F}_0$ for $t \in [-\tau, 0]$. Then the SDDE* (3.1) *has a unique strong solution and this solution is nonnegative. In addition:*

1. *If $\delta = 0$, zero is an absorbing boundary.*
2. *If $\delta > 0$, zero is a reflecting boundary.*
3. *If $\delta < 2$, the process can reach zero and* $\liminf_{t\to\infty} X_t = 0$ *almost surely.*
4. *If $\delta = 2$, zero is inaccessible and X_t reaches arbitrarily small positive values and arbitrarily large values in finite time almost surely.*
5. *If $\delta > 2$, zero is inaccessible and X_t reaches arbitrarily large values in finite time almost surely.*

Note that reflection at the zero boundary is enabled by the positive delay component in the drift term.

We present the proof as we shall use a similar approach for the proposed Memory-Modulated CIR Process with Discrete Delay.

Proof. The existence of a strong solution and the uniqueness of the solution follows from the arguments given in Section 1.1, above.

For the case $\delta = 0$, uniqueness of solutions implies that zero is an absorbing boundary.

For the case $\delta > 0$, we claim that $X_t \geq 0$ and X_t instantaneously reflects at 0. First we use the comparison theorem to prove that the process $X_t \geq 0$. Let X^1 and X^2 be solutions of

$$X_t^1 = \delta X_{t-\tau}^1 dt + 2\sqrt{X_t^1 X_{t-\tau}^1}\, dW_t; \tag{3.2}$$

$$X_t^2 = 2\sqrt{X_t^2 X_{t-\tau}^1}\, dW_t \tag{3.3}$$

where W_t is the same Brownian Motion, $X_t^1 = \theta_t \in [0,\infty)$ for $t \in [-\tau, 0]$ and $X_0^2 = \theta_0$. It is clear that the SDE (3.3) has a unique strong solution due to linear growth and the Yamada-Watanabe Theorem 2.3. It is also clear that $X_t^2 \geq 0$ since $X_t^2 = 0$ is an equilibrium solution and $\theta_0 > 0$. Then arguing inductively over the intervals $[n\tau, (n+1)\tau]$, the comparison theorem (Theorem 2.4) gives us that,

$$P\left[X_t^1 \geq 0 \text{ for all } t \geq 0\right] = 1.$$

Using Theorem 2.5 we have,

$$L_t^0 = 2\delta \int_0^t 1_{\{X_s=0\}} X_{s-\tau}\, ds$$

and we have the differential of the quadratic variation process

$$d\langle X, X\rangle_t = 4X_{s-\tau} X_t\, dt.$$

Suppose $\theta_t > 0$ a.e. then, for almost all $\omega \in \Omega$,

$$\begin{aligned}\int_0^t X_{s-\tau} ds &\geq \int_0^t 1_{\{X_s>0\}} X_{s-\tau}\, ds \\ &= \int_0^t 1_{\{X_s>0\}} X_{s-\tau} \left(4X_{s-\tau}X_s\right)^{-1} d\langle X, X\rangle_s \\ &= \int_0^\infty 1 \cdot (4a)^{-1} L_t^a\, da,\end{aligned}$$

using the Occupation Time Formula (Theorem 2.6). When $\int_0^t X_{s-\tau}\, ds < \infty$ we have $L_t^0 = 2\delta \int_0^t 1_{\{X_s=0\}} X_{s-\tau}\, ds = 0$ to avoid the explosion of the integral at zero and starting at the initial segment we can argue inductively for each lag time to get for Lebesgue measure m,

$$m(\{s > 0 : X_s = 0\}) = 0.$$

Now we consider the boundary behavior for different values of δ. Suppose $0 < \delta < 2$. Suppose that $0 < \theta_0 < n$ and define $\lambda_n = \inf\{t > 0 : X_t = n \text{ or } X_t = 0\}$. First we claim that for almost all ω there exists n such that $\lambda_n = \infty$ or $X_{\lambda_n} = 0$. Let $X_0 = \theta_0$ and suppose that $U_n : [0, \infty) \to \mathbb{R}$ is twice continuously differentiable on $(0, \infty)$. Fix $T > 0$. By the Itô Formula,

$$\begin{aligned}U_n(X_{\lambda_n \wedge T}) = U_n(\theta_0) + \int_0^{\lambda_n \wedge T} &\left[\delta X_{t-\tau} U_n'(X_t) + 2X_t X_{t-\tau} U_n''(X_t)\right] dt \\ &+ \int_0^{\lambda_n \wedge T} \left[2\sqrt{X_t X_{t-\tau}}\, U_n'(X_t)\right] dW_t \quad (3.4)\end{aligned}$$

and thus we solve

$$\delta \frac{dU_n}{dx} + 2x \frac{d^2 U_n}{dx^2} = 0$$

with boundary conditions $U_n(0) = 0, U_n(n) = 1$. Then

$$U_n(x) = \frac{x^{(1-\delta/2)}}{n^{(1-\delta/2)}}.$$

Applying the Optional Stopping Theorem,

$$\mathbb{E}^\theta\left[U_n(X_{\lambda_n \wedge T})\right] = U_n(\theta_0) = \frac{\theta_0^{(1-\delta/2)}}{n^{(1-\delta/2)}},$$

where $\mathbb{E}^\theta$ is the expectation conditioned on the initial segment θ. Since $U_n(x) \geq 0$ for $x \in [0, n]$,

$$\begin{aligned}U_n(\theta_0) &= \lim_{T\to\infty} \mathbb{E}^\theta\left[U_n(X_{\lambda_n \wedge T})\right] \\ &= \mathbb{E}^\theta\left[U_n(X_{\lambda_n})\,;\, \lambda_n < \infty\right] + \lim_{T\to\infty} \mathbb{E}^\theta\left[U_n(X_{\lambda_n \wedge T})\,;\, \lambda_n = \infty\right] \\ &\geq 1 \cdot P^\theta(X_{\lambda_n} = n),\end{aligned}$$

where P^θ is the conditional law of X_t with respect to the initial segment θ and

$$0 \leq \lim_{T\to\infty} \mathbb{E}^\theta\left[U_n(X_{\lambda_n \wedge T})\,;\, \lambda_n = \infty\right] \leq 1$$

since $0 \le U_n \le 1$. Then,

$$P^\theta(X_{\lambda_n} = n) \le U_n(\theta_0)$$

and thus

$$\lim_{n\to\infty} P^\theta(X_{\lambda_n} = n) = 0. \tag{3.5}$$

Also, since

$$P^\theta(X_{\lambda_n} = n) + P^\theta(\lambda_n = \infty \text{ or } X_{\lambda_n} = 0) = 1, \tag{3.6}$$

then

$$P^\theta(\cup_{n=1}^{\infty} \{\lambda_n = \infty \text{ or } X_{\lambda_n} = 0\}) = \lim_{n\to\infty} P^\theta(\lambda_n = \infty \text{ or } X_{\lambda_n} = 0) = 1, \tag{3.7}$$

which proves the claim.

Now we show that $\liminf_{t\to\infty} X_t(\omega) = 0$ for a.a ω. Let $V_n : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable. Then, again by the Itô formula,

$$\begin{aligned} V_n(X_{\lambda_n \wedge T}) = V_n(\theta_0) &+ \int_0^{\lambda_n\wedge T} [\delta X_{t-\tau} V_n'(X_t) + 2X_t X_{t-\tau} V_n''(X_t)]\, dt \\ &+ \int_0^{\lambda_n\wedge T} \left[2\sqrt{X_t X_{t-\tau}}\, V_n'(X_t)\right] dW_t. \end{aligned} \tag{3.8}$$

We factor out $X_{t-\tau}$ from the generator to construct the second order differential equation

$$\delta \frac{dV_n}{dx} + 2x \frac{d^2 V_n}{dx^2} = -1$$

which has solution $V_n(x) = (n-x)/\delta$ satisfying boundary condition $V_n(n) = 0$ and for which $V_n(x) \ge 0$ on $[0, n]$. Then

$$\lim_{T\to\infty} \mathbb{E}^\theta [V_n(X_{\lambda_n\wedge T})] - V_n(\theta_0) = -\lim_{T\to\infty} \mathbb{E}^\theta \left[\int_0^{\lambda_n\wedge T} X_{t-\tau}\, dt\right].$$

Thus

$$\lim_{T\to\infty} \mathbb{E}^\theta \left[\int_0^{\lambda_n\wedge T} X_{t-\tau}\, dt\right] \le V_n(\theta_0) < \infty;$$

and hence

$$\mathbb{E}^\theta \left[\int_0^{\lambda_n} X_{t-\tau}\, dt\, ;\, \lambda_n < \infty\right] + \mathbb{E}^\theta \left[\int_0^{\infty} X_{t-\tau}\, dt\, ;\, \lambda_n = \infty\right] < \infty$$

so that

$$\mathbb{E}^\theta \left[\int_0^{\lambda_n} X_{t-\tau}\, dt\, ;\, \lambda_n = \infty\right] < \infty.$$

If there is a n such that $P^\theta(\lambda_n = \infty) > 0$, then on that event, $\liminf_{t\to\infty} X_t = 0$. Otherwise X_t reaches zero and reflects instantaneously. Since the same is true if the process is considered for $t \in [S, \infty)$, for arbitrarily large time S, $\liminf_{t\to\infty} X_t = 0$ a.s.

Now consider the case $\delta = 2$. Suppose that $0 < \epsilon < \theta_0 < n$. Define

$$\lambda_{\epsilon n} = \inf\{t > 0 : X_t = \epsilon \text{ or } X_t = n\}.$$

Again, $V_n(x) = (n-x)/2$ is a solution of $2V_n'(x) + 2xV_n''(x) = -1$, with boundary condition $V_n(n) = 0$. Hence,

$$\begin{aligned}\mathbb{E}^\theta\left[\int_0^T X_{t-\tau}\,dt\,;\,\lambda_{\epsilon n} = \infty\right] &= V_n(\theta_0) - \mathbb{E}^\theta\left[V_n(X_{\lambda_{\epsilon n}\wedge T})\right] \\ &\qquad - \mathbb{E}^\theta\left[\int_0^{\lambda_{\epsilon n}\wedge T} X_{t-\tau}\,dt\,;\,\lambda_{\epsilon n} < \infty\right] \\ &\le V_n(\theta_0) < \infty.\end{aligned}$$

Since $X_t > \epsilon$ for all t when $\lambda_{\epsilon n} = \infty$, letting $T \to \infty$, it follows that $P^\theta(\lambda_{\epsilon n} = \infty) = 0$.

Now let

$$U_{\epsilon n}(x) = \frac{\log(x/\epsilon)}{\log(n/\epsilon)}.$$

Then $U_{\epsilon n}'(x) + xU_{\epsilon n}''(x) = 0$, $U_{\epsilon n}(\epsilon) = 1$, and $U_{\epsilon n}(n) = 0$. Thus

$$U_{\epsilon n}(\theta_0) = \lim_{T\to\infty} \mathbb{E}^\theta\left[U_{\epsilon n}(X_{\lambda_{\epsilon n}\wedge T})\right] = P^\theta(X_{\lambda_{\epsilon n}} = n).$$

Therefore $\lim_{\epsilon\to 0} P^\theta(X_{\lambda_{\epsilon n}} = n) = 1$ for every n. Since this continuous process almost surely reaches every positive integer before it hits zero, it almost surely cannot reach zero in finite time.

Arguing in a similar way, let

$$\tilde{U}_{\epsilon n}(x) = \frac{\log(n/x)}{\log(n/\epsilon)}.$$

Thus

$$\tilde{U}_{\epsilon n}(\theta_0) = \lim_{T\to\infty} \mathbb{E}^\theta\left[\tilde{U}_{\epsilon n}(X_{\lambda_{\epsilon n}\wedge T})\right] = P^\theta(X_{\lambda_{\epsilon n}} = \epsilon).$$

In this case, $\lim_{n\to\infty} P^\theta(X_{\lambda_{\epsilon n}} = \epsilon) = 1$, which proves that the process almost surely reaches every positive $\epsilon < \theta_0$ in finite time.

Now suppose $\delta > 2$. We define $\mu_m = \inf\{t > 0 : X_t = 1/m\}$ and solve

$$\delta\frac{dU_m}{dx} + 2x\frac{d^2U_m}{dx^2} = 0,$$

with boundary condition $U_m(1/m) = 1$, to get

$$U_m(x) = \frac{1}{x^{(\delta/2-1)}m^{(\delta/2-1)}}.$$

With an argument similar to the case where $\delta < 2$, we get

$$P^\theta(\cup_{m=1}^\infty \{\mu_m = \infty\}) = \lim_{m\to\infty} P^\theta\{\mu_m = \infty\} = 1 \tag{3.9}$$

and thus X_t is almost surely bounded away from 0.

Now we prove that X_t reaches any $n > \theta_0$ a.s. when $\delta > 2$. Define $\lambda_n = \inf\{t > 0 : X_t = n\}$. Then by solving,

$$\delta\frac{dV_n}{dx} + 2x\frac{d^2V_n}{dx^2} = -1,$$

with boundary condition $V_n(n) = 0$, we get $V_n = (n-x)/\delta$ and hence, by an argument similar to before,

$$\mathbb{E}^{\theta}\left[\int_0^{\lambda_n} X_{t-\tau}\,dt\right] \le V_n(\theta_0) < \infty.$$

Then,

$$\mathbb{E}^{\theta}\left[\int_0^{\lambda_n} X_{t-\tau}\,dt\,;\,\lambda_n < \infty\right] + \mathbb{E}^{\theta}\left[\int_0^{\infty} X_{t-\tau}\,dt\,;\,\lambda_n = \infty\right] < \infty.$$

Arguing by contradiction, suppose that $P^{\theta}(\lambda_n = \infty) > 0$. Then using (3.9) there exists $N \in \mathbb{N}$ s.t. $P^{\theta}(\mu_N = \infty, \lambda_n = \infty) > 0$ and

$$\mathbb{E}^{\theta}\left[\int_0^{\infty} X_{t-\tau}\,dt;\,\lambda_n = \infty\right] = \infty$$

which leads to a contradiction. Therefore, $P^{\theta}(\lambda_n = \infty) = 0$. □

3.2. Memory-modulated CIR process with discrete delay. We use a analysis similar to previous section in the preceding section to derive the following results for our proposed Memory-Modulated CIR Process with Discrete Delay.

Theorem 3.2. *Let τ, K, and α be positive constants and γ be a nonnegative constant. Let $\beta = 2\gamma K/\alpha^2$. Suppose that θ_t is positive, integrable, and independent of $\mathcal{F}_0$ for $t \in [-\tau, 0]$. Then the SDDE*

$$dX_t = \gamma X_{t-\tau}\left(K - X_t\right)dt + \alpha\sqrt{X_t X_{t-\tau}}\,dW_t; \quad X_t = \theta_t \text{ for } t \in [-\tau, 0] \tag{3.10}$$

has a unique strong solution and this solution is nonnegative. In addition:

1. *If $\gamma = 0$ (and hence $\beta = 0$), then zero is an absorbing boundary.*
2. *If $\beta > 0$, zero is a reflecting boundary.*
3. *If $0 < \beta < 1$, then X_t can reach zero and $\liminf_{t\to\infty} X_t = 0$ almost surely.*
4. *If $\beta \ge 1$, then zero is inaccessible and X_t reaches arbitrarily small positive values and arbitrarily large values in finite time almost surely.*

Again the proof of the existence and uniqueness follows from the comments in Section 1.1 and we will prove the boundary behavior using the generator of the process.

Proof. Note that in this proof we use some of the same notation from the previous proof while some being redefined. First we use the comparison theorem to prove that the process $X_t \ge 0$. Let

$$X_t^1 = \gamma X_{t-\tau}^1\left(K - X_t^1\right)dt + \alpha\sqrt{X_t^1 X_{t-\tau}^1}\,dW_t; \tag{3.11}$$

$$X_t^2 = \alpha\sqrt{X_t^2 X_{t-\tau}^1}\,dW_t \tag{3.12}$$

where W_t is the same Brownian motion, $X_t^1 = \theta_t \in [0, \infty)$ for $t \in [-\tau, 0]$ and $X_0^2 = \theta_0$ or is started at $X_{t_0}^2 = K$ for a time t_0 as described below. It is clear that the SDE (3.12) has a unique strong solution due to linear growth and Theorem 2.3. It is also clear that $X_t^2 \ge 0$ since $X_t^2 = 0$ is an equilibrium solution and $\theta_0 \ge 0$. Then arguing inductively over the intervals $[n\tau, (n+1)\tau]$ and starting X^2

at $X^2_{t_0} = K$ for each time t_0 when the process X^1 reenters the interval $(-\infty, K]$, the comparison theorem (Theorem 2.4) gives us that,

$$P\left[X^1_t \geq 0 \text{ for all } t \geq 0\right] = 1.$$

Then with a similar argument as in the proof of Theorem 3.2 we can prove that the process reflects at zero instantaneously when $\beta > 0$. Now we prove the boundary behavior of the process.

Now suppose $0 < \beta < 1$. Define: $\lambda_n = \inf\{t > 0 : X_t = n \text{ or } X_t = 0\}$. Now we claim that for a.a. ω there exist n such that $\lambda_n = \infty$ or $X_{\lambda_n} = 0$. Let $X_0 = \theta_0$ and $U_n : [0, \infty) \to \mathbb{R}$ be twice continuously differentiable on $(0, \infty)$. Then for any $T > 0$, using the Itô formula, we can write,

$$\begin{aligned} &U_n(X_{\lambda_n \wedge T}) \\ &= U_n(\theta_0) + \int_0^{\lambda_n \wedge T} \left[\gamma X_{t-\tau}\left(K - X_t\right) U_n'(X_t) + \frac{1}{2}\alpha^2 X_t X_{t-\tau} U_n''(X_t)\right] dt \\ &\qquad + \int_0^{\lambda_n \wedge T} \left[\alpha\sqrt{X_t X_{t-\tau}}\, U_n'(X_t)\right] dW_t \end{aligned} \tag{3.13}$$

and we solve

$$\gamma\left(K - x\right)\frac{dU_n}{dx} + \frac{1}{2}\alpha^2 x \frac{d^2 U_n}{dx^2} = 0$$

with boundary conditions $U_n(0) = 0, U_n(n) = 1$. Then

$$U_n(x) = \frac{\int_0^x \exp(\beta y/K) y^{-\beta}\, dy}{\int_0^n \exp(\beta y/K) y^{-\beta}\, dy}. \tag{3.14}$$

and

$$\mathbb{E}^\theta\left[U_n(X_{\lambda_n \wedge T})\right] = U_n(\theta_0),$$

where, as before, $\mathbb{E}^\theta$ is the expectation conditioned on the initial segment $X_t = \theta_t$ for $t \in [-\tau, 0]$. Since $U_n(x) \geq 0$ for $x \geq 0$,

$$\begin{aligned} U_n(\theta_0) &= \lim_{T\to\infty} \mathbb{E}^\theta\left[U_n(X_{\lambda_n \wedge T})\right] \\ &= \mathbb{E}^\theta\left[U_n(X_{\lambda_n})\,;\, \lambda_n < \infty\right] + \lim_{T\to\infty} \mathbb{E}^\theta\left[U_n(X_{\lambda_n \wedge T})\,;\, \lambda_n = \infty\right] \\ &\geq 1 \cdot P^\theta(X_{\lambda_n} = n) + 0, \end{aligned}$$

where P^θ is the law of X_t conditioned on the initial segment θ. Since $\lim_{n\to\infty} U_n(x)$ equals zero,

$$\lim_{n\to\infty} P^\theta(X_{\lambda_n} = n) = 0. \tag{3.15}$$

We also have that

$$P^\theta(X_{\lambda_n} = n) + P^\theta(\lambda_n = \infty \text{ or } X_{\lambda_n} = 0) = 1$$

and therefore

$$P^\theta(\cup_{n=1}^\infty \{\lambda_n = \infty \text{ or } X_{\lambda_n} = 0\}) = \lim_{n\to\infty} P^\theta(\lambda_n = \infty \text{ or } X_{\lambda_n} = 0) = 1$$

which proves the claim.

Now we prove that $\liminf_{t\to\infty} X_t(\omega) = 0$ for a.a ω. Let $V_n : [0,\infty) \to \mathbb{R}$ be twice continuously differentiable. Then,

$$\begin{aligned} &V_n(X_{\lambda_n\wedge T}) \\ &= V_n(\theta_0) + \int_0^{\lambda_n\wedge T} \left[\gamma X_{t-\tau}(K - X_t)V_n'(X_t) + \frac{1}{2}\alpha^2 X_t X_{t-\tau} V_n''(X_t)\right] dt \\ &\qquad + \int_0^{\lambda_n\wedge T} \left[\alpha\sqrt{X_t X_{t-\tau}} V_n'(X_t)\right] dW_t. \end{aligned} \tag{3.16}$$

We split off the delay factor in the generator to construct and solve the following second order differential equation.

$$\gamma(K - x)\frac{dV_n}{dx} + \frac{1}{2}\alpha^2 x\frac{d^2V_n}{dx^2} = -1$$

with boundary conditions $V_n(0) = V_n(n) = 0$. Then

$$V_n(x) = \frac{1}{\gamma}\left(\frac{K}{\beta}\right)^{\beta-1}\int_x^n (\Gamma(\beta, \beta t/K) - C_n)\, t^{-\beta}\exp(\beta t/K)\, dt,$$

where $\Gamma(\beta, x)$ is the incomplete Gamma function,

$$\Gamma(\beta, x) = \int_0^x y^{\beta-1}e^{-y}\, dy,$$

and the constant $C_n > 0$ is chosen so that $V_n(0) = 0$. Then $V_n(x) > 0$ for $0 < x < n$. We have then that

$$0 \le \lim_{T\to\infty} \mathbb{E}^\theta\left[V_n(X_{\lambda_n\wedge T})\right] = V_n(\theta_0) - \lim_{T\to\infty}\mathbb{E}^\theta\left[\int_0^{\lambda_n\wedge T} X_{t-\tau}\, dt\right]$$

and thus

$$\mathbb{E}^\theta\left[\int_0^\infty X_{t-\tau}\, dt\,;\, \lambda_n = \infty\right] \le \lim_{T\to\infty}\mathbb{E}^\theta\left[\int_0^{\lambda_n\wedge T} X_{t-\tau}\, dt\right] \le \mathbb{E}^\theta\left[V_n(\theta_0)\right] < \infty.$$

It follows that if there exists an n such that $P^\theta(\lambda_n = \infty) > 0$, then on that event $\liminf_{t\to\infty} X_t = 0$. Otherwise X_t reaches zero and reflects instantaneously. As we noted before, since the same is true if the process is considered for $t \in [S,\infty)$, for arbitrarily large time S, $\liminf_{t\to\infty} X_t = 0$ a.s.

Now consider the case $\beta \ge 1$. Suppose that $0 < \epsilon < \theta_0 < n$. Define

$$\lambda_{\epsilon n} = \inf\{t > 0 : X_t = \epsilon \text{ or } X_t = n\}.$$

As before,

$$V_n(x) = \frac{1}{\gamma}\left(\frac{K}{\beta}\right)^{\beta-1}\int_x^n (\Gamma(\beta, \beta t/K) - C_n)\, t^{-\beta}\exp(\beta t/K)\, dt, \tag{3.17}$$

is a solution of

$$\gamma(K - x)\frac{dV_n}{dx} + \frac{1}{2}\alpha^2 x\frac{d^2V_n}{dx^2} = -1$$

and satisfies the boundary condition $V_n(n) = 0$. When $\beta = 1$, letting $C_n = 0$, this solution reduces to

$$V_n(x) = \int_x^n \frac{e^{t/K} - 1}{\gamma t}\, dt. \tag{3.18}$$

and satisfies $0 \le V_n(x) < \infty$ for $0 \le x \le n$. When $\beta > 1$, given $\epsilon > 0$, we may choose the constant C_n in (3.17) so that $V_n(\epsilon) = 0$ and $V_n(x) > 0$ for $\epsilon < x < n$. In either case, it follows that

$$\begin{aligned}\mathbb{E}^\theta \left[\int_0^T X_{t-\tau}\, dt\, ;\, \lambda_{\epsilon n} = \infty \right] &= V_n(\theta_0) - \mathbb{E}^\theta \left[V_n(X_{\lambda_{\epsilon n} \wedge T}) \right] \\ &\quad - \mathbb{E}^\theta \left[\int_0^{\lambda_{\epsilon n} \wedge T} X_{t-\tau}\, dt\, ;\, \lambda_{\epsilon n} < \infty \right] \\ &\le V_n(\theta_0) < \infty.\end{aligned}$$

Since $X_t > \epsilon$ for all t when $\lambda_{\epsilon n} = \infty$, letting $T \to \infty$, it follows that $P^\theta(\lambda_{\epsilon n} = \infty) = 0$.

Now let

$$U_{\epsilon n}(x) = \frac{\int_\epsilon^x \exp(\beta y/K) y^{-\beta}\, dy}{\int_\epsilon^n \exp(\beta y/K) y^{-\beta}\, dy}. \tag{3.19}$$

which satisfies $U_{\epsilon n}(\epsilon) = 0$ and $U_{\epsilon n}(n) = 1$. Then

$$U_{\epsilon n}(\theta_0) = \lim_{T \to \infty} \mathbb{E}^\theta \left[U_{\epsilon n}(X_{\lambda_{\epsilon n} \wedge T}) \right] = P^\theta(X_{\lambda_{\epsilon n}} = n).$$

Therefore $\lim_{\epsilon \to 0} P^\theta(X_{\lambda_{\epsilon n}} = n) = 1$ for every n. Since this continuous process almost surely reaches every positive integer before it hits zero, it almost surely cannot reach zero in finite time.

Arguing in a similar way, let

$$\tilde{U}_{\epsilon n}(x) = \frac{\int_x^n \exp(\beta y/K) y^{-\beta}\, dy}{\int_\epsilon^n \exp(\beta y/K) y^{-\beta}\, dy}. \tag{3.20}$$

Thus $\tilde{U}_{\epsilon n}(\epsilon) = 1$ and $\tilde{U}_{\epsilon n}(n) = 0$ and

$$\tilde{U}_{\epsilon n}(\theta_0) = \lim_{T \to \infty} \mathbb{E}^\theta \left[\tilde{U}_{\epsilon n}(X_{\lambda_{\epsilon n} \wedge T}) \right] = P^\theta(X_{\lambda_{\epsilon n}} = \epsilon).$$

Therefore, $\lim_{n \to \infty} P^\theta(X_{\lambda_{\epsilon n}} = \epsilon) = 1$ for every ϵ, which proves that the process almost surely reaches every positive $\epsilon < \theta_0$ in finite time. □

3.3. Remarks. We note that in the $\delta > 2$ case for the (memory-modulated) Squared Bessel Process with Discrete Delay, the process exhibits a kind of transience where the process is ultimately bounded away from zero with probability one. This does not occur for large β in the Memory-Modulated CIR Process with Discrete Delay where due to the limiting factor $K - X_t$ in the drift term, when $\beta \ge 1$, the process returns arbitrarily close to zero in finite time. This behavior is like that in the critical case $\delta = 2$ for the Squared Bessel Process with Discrete Delay.

In these latter cases ($\delta = 2$ and $\beta \ge 1$), if $0 < \epsilon < \theta_0 < n < \infty$, then the exit time from the interval $[\epsilon, n]$ is almost surely finite and the probability (conditioned on the initial segment θ) of exit at endpoint n or ϵ is determined explicitly as $U_{\epsilon n}(\theta_0)$ or $\tilde{U}_{\epsilon n}(\theta_0)$, respectively.

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