Matrices over differential fields which commute with their derivative

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A theorem is proved concerning the diagonalizability of a matrix over a differential field by means of a similarity transformation from the field of constants of the differential field. This result contains, as a special case, known results concerning the diagonalizability over the complex numbers of a Hermitian matrix of analytic functions under the hypothesis that the matrix commutes with its derivative.

1. INTRODUCTION

If $K$ is a field and $F$ is a subfield, then we will say that a matrix $A \in M_n(K)$ is diagonalizable over $F$ if there is an invertible matrix $P \in GL(n, F)$ such that $P^{-1}AP$ is diagonal. Of course, this just means that $A$ is
diagonalizable (over $K$) and that each eigenspace $V \subseteq K^n$ has a basis consisting of vectors from $F^n$. We will be concerned with the problem of diagonalizing a matrix with entries from a differential field by means of a similarity over the field of constants. Before stating the main result, we will recall the definition of differential field. An (ordinary) differential field is a field $K$ together with a derivation (which we will denote by $a \mapsto a'$). The field of constants $F$ of $K$ is the set of elements of $K$ with derivative 0. If $K$ is a differential field and $A \in M_n(K)$, then $A' \in M_n(K)$ will denote the matrix obtained by differentiating each element of $A$. For the theory of differential fields, one may consult Kolchin [9] or Kaplansky [8]. We simply note that the theory of differential fields was developed (primarily by Ritt and Kolchin) to provide an algebraic setting for the theory of differential equations with meromorphic coefficients. However, we will need no deep results from the theory of differential fields.

The main theorem to be proved is the following.

**Theorem.** Let $K$ be an ordinary differential field with field of constants $F$. If $A \in M_n(K)$ is diagonalizable over $K$ and $[A, A'] = 0$, i.e., $A$ commutes with $A'$, then $A$ is diagonalizable over $F$.

Results similar to this theorem have been proved for various rings of functions by a number of authors; Evard [3] has an extensive bibliography.

2. MAIN THEOREM

**Lemma 2.1.** Let $K$ be a differential field, and let $A \in M_n(K)$ be a matrix such that $[A, A'] = 0$. Suppose that

$$K^n = V_1 \oplus \cdots \oplus V_m$$

is the primary decomposition of $K^n$ determined by $A$ [6, p. 220]. Then $V_i$ is a differential subspace of $K^n$ for each $i$, i.e., $V_i$ is invariant under differentiation.

**Proof.** Let $P_i$ denote the projection of $K^n$ onto $V_i$ determined by the decomposition (2.1). According to the primary decomposition theorem, $P_i = q_i(A)$ for some polynomial $q_i(X) \in K[X]$. Since $[A, A'] = 0$, it follows that

$$[P_i, P_i'] = 0.$$
We claim that $P_i' = 0$. Indeed, $P_i^2 = P_i$, so Equation (2.2) implies that
$P_i' = 2P_i'$. Hence

$$P_iP'_i = 2P_i^2P'_i = 2P_iP'_i,$$

so that $P_iP'_i = 0$. Thus $P_i' = 0$, as desired.

Now let $v \in V_i$. Then $v = P_iv$, so

$$v' = (P_iv)' = P'_iv + P_i'v \in V_i.$$

**COROLLARY 2.2.** Let $K$ be a differential field, and let $A \in M_n(K)$. Suppose that $A$ is diagonalizable and $[A, A'] = 0$. If $V \subseteq K^n$ is the eigenspace of $A$ corresponding to the eigenvalue $\mu$, then $V$ is a differential subspace of $K^n$.

**Proof.** Since $A$ is diagonalizable, the primary components of $A$ are the eigenspaces.

If $R$ is a ring and $\Gamma$ is an index set, then $R^\Gamma$ denotes the free $R$-module on the index set $\Gamma$. An $R$-basis of $R^\Gamma$ is $\{\delta_\alpha\}_{\alpha \in \Gamma}$, where $\delta_\alpha$ is 1 at the $\alpha$th coordinate, and 0 elsewhere.

**LEMMA 2.3.** Let $K$ be a differential field with field of constants $F$. Let $S$ be a differential subspace of the differential vector space $K^\Gamma$. Then $S$ contains a $K$-basis consisting of elements of $F^\Gamma$.

**Proof.** Consider the skew polynomial ring $T = K[X, ']$. That is, the elements of $T$ are polynomials in $X$ with coefficients in $K$, and the multiplication on $T$ is determined by the formula

$$Xa = aX + a',$$

where $a'$ denotes the derivative of $a$ in the differential field $K$. The differential field $K$ becomes a left $T$-module by means of the scalar multiplication $Xv = v'$, and hence $K^\Gamma$ is also a left $T$-module by means of coordinatewise multiplication by elements of $T$.

Note that $K$ is a simple $T$-module (since it is already simple as a $K$-module) and hence $K^\Gamma$ is a semisimple $T$-module, with all the simple components being equal to $K$. A differential subspace $S$ of $K^\Gamma$ is a $T$-submodule of the $T$-module $K^\Gamma$, and hence it is isomorphic to a direct sum of simple components of $K^\Gamma$, i.e., $S$ is isomorphic as a $T$-module to $K^\Lambda$ for
some index set $\Lambda \subseteq \Gamma$. Let $\psi: K^\Lambda \to S$ be a $T$-module isomorphism, and let

$$\{v_\alpha\}_{\alpha \in \Lambda} = \{\psi(\delta_\alpha)\}_{\alpha \in \Lambda}$$

be the image under $\psi$ of the standard $K$-basis on the free $K$-module $K^\Lambda$. Since $\psi$ is a $T$-module homomorphism, it follows that

$$v'_\alpha = Xv_\alpha = X\psi(\delta_\alpha) = \psi(X\delta_\alpha) = \psi(\delta'_\alpha) = 0.$$  

Hence $v_\alpha \in F^\Gamma$, and the proof is complete. 

The main theorem now follows immediately.

**Theorem 2.4.** Let $K$ be a differential field with field of constants $F$, and let $A \in M_n(K)$. If $A$ is diagonalizable over $K$ and $[A, A'] = 0$, then $A$ is diagonalizable over $F$.

**Proof.** Since $A$ is diagonalizable, $K^n = \bigoplus_{i=1}^r V_i$, where $V_i$ is an eigenspace of $A$ with eigenvalue $\mu_i$, and $\mu_i \neq \mu_j$ if $i \neq j$. By Corollary 2.2, each $V_i$ is a differential subspace of $K^n$, and Lemma 2.3 shows that $V_i$ has a basis of vectors from $F^n$, which is what we wished to show.

**Remark 2.5.** Suppose that $K$ is a field and $A \in M_n(K)$ is a matrix which is diagonalizable over a subfield $F \subseteq K$. If $\alpha$ and $\beta$ are two places of $K$ defined over $F$ and with the same residue field $L$, then $A$ determines two matrices $A(\alpha)$ and $A(\beta) \in M_n(L)$ by evaluating the places $\alpha$ and $\beta$ on each element of $A$, provided the places are finite at every element of $A$. Since there is $P \in GL(n, F)$ such that $P^{-1}AP = D$ where $D$ is diagonal, it follows that $\alpha$ and $\beta$ are both finite on $D$, and hence, $P^{-1}A(\alpha)P = D(\alpha)$ and $P^{-1}A(\beta)P = D(\beta)$. Therefore, $A(\alpha)$ and $A(\beta)$ are commutative. This is an algebraic version of functional commutativity for matrices with entries in a ring of functions. As a particular case, this means that a matrix $A$ satisfying the hypotheses of Theorem 2.4 is functionally commutative.

**Remark 2.6.** By applying Lemma 2.1 and Lemma 2.3 to the general primary decomposition, rather than to the special case of a diagonalizable matrix, one can show that any matrix $A \in M_n(K)$ such that $[A, A'] = 0$ is similar over the field of constants $F$ to a block diagonal matrix $A_1 \oplus \cdots \oplus A_m$, where $A_i \in M_{d_i}(K)$, with $d_i$ being the dimension of the $i$th primary component $V_i$, and $A_i$ annihilated by the minimal polynomial of $A_i|_{V_i}$. 

We conclude this section with some observations concerning diagonalizable matrices which commute with their derivative.

**Lemma 2.7.** Let \( K \) be a field, and let \( A, B \in M_n(K) \) be matrices such that \( A \) is diagonalizable over \( K \) and \([A, [A, B]] = 0\). Then \([A, B] = 0\).

**Proof.** Let \( \text{Ad}_A(C) = [A, C] \). Then \( \text{Ad}_A : M_n(A) \to M_n(A) \) is a linear transformation. Since \( A \) is diagonalizable, it follows that \( \text{Ad}_A \) is diagonalizable [7, p. 181]. Thus \( (\text{Ad}_A)^2(B) = 0 \) implies that \([A, B] = \text{Ad}_A(B) = 0\). \( \blacksquare \)

**Lemma 2.8.** Let \( K \) be a differential field, and let \( A \in M_n(K) \) be diagonalizable (over \( K \)). If \([A, A'] = 0\), then \( A' \) is also diagonalizable. In fact, if \( P^{-1}AP = D \), where \( D \in M_n(K) \) is a diagonal matrix, then \( P^{-1}AP' = D' \).

**Proof.** Assume that \( A = PDP^{-1} \), where \( P \in \text{GL}(n, K) \) and \( D \) is diagonal. Then differentiating \( PP^{-1} = I_n \) gives

\[
P(P^{-1})' + P'P^{-1} = 0.
\]

so that

\[
(P^{-1})' = -P^{-1}P'P^{-1}.
\]

Then

\[
A' = (PDP^{-1})'
\]

\[
= P'(DP^{-1} + PD'P^{-1} - PDP^{-1}P'P^{-1}) \quad (2.3)
\]

If we let \( B = P^{-1}P' \), then Equation (2.3) gives

\[
A' = P( BD + D' - DB)P^{-1}
\]

\[
= P([B, D] + D')P^{-1}. \quad (2.4)
\]

But \([A, A'] = 0\), so we obtain

\[
0 = [A, A']
\]

\[
= [PDP^{-1}, P([B, D] + D')P^{-1}]
\]

\[
= P[D, [B, D] + D']P^{-1}. \quad (2.5)
\]
Thus $[D, [D, B]] = -[D, [B, D]] = 0$, so Lemma 2.7 shows that $[B, D] = 0$ and Equation (2.4) shows that $A' = PD'P^{-1}$.

The following result is an immediate corollary of Lemma 2.8.

**Corollary 2.9.** Let $K$ be a differential field, and let $A \in M_n(K)$ be diagonalizable (over $K$). If $A$ commutes with $A'$, then $A'$ is a polynomial in $A$. In other words, $K[A]$ is a differential subalgebra of $M_n(K)$.

**Proof.** By Lemma 2.8, there is $P \in GL(n, K)$ with $P^{-1}AP = D$ and $P^{-1}A'P = D'$, where $D = \text{diag}(a_1, \ldots, a_n)$ is a diagonal matrix. Let $f(X) \in K[X]$ be any polynomial such that $f(a_i) = a'_i$. Then $f(A) = A'$.

### 3. Examples and Consequences

Theorem 2.4 contains, as special cases, some previous results:

**Example 3.1.** Let $I$ be a closed interval in $\mathbb{R}$, and let $R$ denote the ring of germs of complex analytic functions on a neighborhood of $I$. Now $R$ is an integral domain, so let $K$ be the quotient field of $R$. Ordinary differentiation of complex analytic functions along $I$ induces a derivation on $K$ with field of constants the complex numbers $\mathbb{C}$. Suppose that $A \in M_n(R)$ is Hermitian. Then Rellich's theorem [10] shows that $A$ is diagonalizable over $R$. Since $A'$ is also Hermitian, one obtains the following theorem of Goff [5] as a special case of Theorem 2.4.

**Theorem 3.2.** If $A \in M_n(R)$ is Hermitian and $AA' = A'A$, then $A$ is diagonalizable over $\mathbb{C}$.

**Example 3.3.** If $R$ is a real closed field and $C = R[i]$ is the algebraic closure, then one has a natural conjugation involution on $C[[t]]$, and hence it is meaningful to speak of normal matrices with entries in $C[[t]]$ (see [2]). According to [2], normal matrices with entries in $C[[t]]$ can be unitarily diagonalized over $C[[t]]$. Note that differentiation with respect to $t$ is a derivation on $C[[t]]$ which extends naturally to the quotient field $K$ with field of constants equal to $C$. Thus Theorem 2.4 can be applied to this situation.

**Theorem 3.4.** Let $A \in M_n(C[[t]])$ be a normal matrix. If $AA' = A'A$, then $A$ can be unitarily diagonalized over $C$.

The above result was presented without details in [2].
While it is not needed for the application of Theorem 2.4 to Theorem 3.4, the following elementary fact is of interest. It can be thought of as a special case of Lemma 2.8.

**Lemma 3.5.** Let $A \in M_n(C[[t]])$ be a normal matrix which commutes with its derivative $A'$. Then $A'$ is normal.

*Proof.* Since $A$ is normal, it can be unitarily diagonalized over $C[[t]]$ (see [2, Theorem 4.5]). A standard argument for normal matrices over the complex numbers applies to show that the Hermitian transpose $A^*$ is a polynomial in $A$ with coefficients from the quotient field $K$ of $C[[t]]$. That is, $A^* \in K[A]$. Since $AA' = A'A$, Corollary 2.9 shows that $K[A]$ is a commutative differential subalgebra of $M_n(K)$. Then $(A')^* = (A^*)' \in K[A]$, and hence $A'$ commutes with $(A')^*$.

There is one other situation to which Theorem 2.4 naturally applies. Suppose that $k$ is an algebraically closed field of characteristic 0 and $K$ is an algebraic function field over $k$ of dimension 1. Then $K$ is the field of rational functions of a nonsingular projective algebraic curve $X$ over $k$. If $A \in M_n(K)$, then one can evaluate $A$ at each $p \in X$ to get $A(p) \in M_n(k)$ (see [1]). We say that $A$ is pointwise diagonalizable if $A(p)$ is diagonalizable over $k$ for all $p \in X$. Any nontrivial derivation of $K$ over $k$ will make $K$ into a differential field with field of constants $k$. With this notation we get the following result.

**Theorem 3.6.** With the above notation, if $A$ is pointwise diagonalizable and if $A$ and $A'$ commute, then $A$ is diagonalizable over $k$.

*Proof.* By Lemma 3.2 of [1], $A$ is diagonalizable over a finite extension field $L$ of $K$. Since the derivation of $K$ extends uniquely to $L$, we are in the situation of Theorem 2.4, and hence $A$ is diagonalizable over $k$.

*Remark.* Theorem 3.6 should be compared with Theorem 3.3 of [1]. The latter theorem shows that $A \in M_n(K)$ is diagonalizable over $k$ provided that $A$ is pointwise diagonalizable and the eigenvalues of $A$ do not agree to a "high order" at any point (see [1] for the precise statement). The present result replaces the eigenvalue condition with commutativity of $A$ and its derivative.

We conclude with the following observations:

**Lemma 3.7.** Let $K$ be a differential field with field of constants $F$, and let $C$ be a differential subalgebra of $M_n(K)$. Then there is a subalgebra $G \subseteq M_n(F)$ such that $C = KG$.

*Proof.* This is just a special case of Lemma 2.3.
Corollary 3.8. Let $K$ be a differential field with field of constants $F$, and let $A \in M_n(K)$ be a matrix such that $A' \in K[A]$. Then $A = q(B)$ where $B \in M_n(F)$ and $q(X) \in K[X]$.

Proof. We will consider separately the following two cases.

Case 1. $|F| < \infty$. In this case, $\text{char}(F) = \text{char}(K) = p < \infty$, so that $K$ has as prime subfield the field $F_p$ with $p$ elements. Since $\text{char}(K) = p$, it follows that $a' = 0$ for all $p$th powers $a \in K$, i.e., $K^p \subseteq F$. If $x \in K$ is transcendental over $F_p$, then $F_p[x^p] \subseteq K^p \subseteq F$, so $|F| = \infty$. Hence, $K$ is algebraic over $F_p$. Thus $K$ is perfect and $K = F$.

Case 2. $|F| = \infty$. Consider the algebra $K[A] \subseteq M_n(K)$. By hypothesis, this is a differential subalgebra of $M_n(K)$, and Lemma 3.7 shows that $K[A] = KG$ where $G$ is a subalgebra of $M_n(F)$. Let

$$d = \dim_F G = \dim_K K[A] = \deg m_A(X),$$  \hspace{1cm} (3.1)

where $m_A(X)$ is the minimal polynomial of $A$ (over $K$). If $L$ is any field and $C \in M_n(L)$, then there is a rank criterion for the degree of $m_C(X)$. Namely,

$$\deg m_C(X) = \text{rank}_L M(C),$$  \hspace{1cm} (3.2)

where $M(C)$ is the $n^2 \times n$ matrix whose $i$th column is $C^{i-1}$ considered as a column vector of size $n^2$.

Now let $L = K(X_1, \ldots, X_d)$ be the field of rational functions over $K$ in the indeterminants $X_1, \ldots, X_d$, let $A_1, \ldots, A_d$ be a basis of $G$ over $F$, and let

$$C = X_1A_1 + \cdots + X_dA_d \in M_n(F(X_1, \ldots, X_d)) \subseteq M_n(L).$$  \hspace{1cm} (3.3)

If $a = (a_1, \ldots, a_d) \in K^d$, then the specialization of $C$ at $a$ is

$$C(a) = a_1A_1 + \cdots + a_dA_d \in KG = K[A].$$  \hspace{1cm} (3.4)

Since $F$ (and hence $K$) is infinite,

$$\text{rank}_L M(C) = \max_{a \in K^d} \text{rank}_K M(C(a)) = \max_{a \in F^d} \text{rank}_F M(C(a)).$$  \hspace{1cm} (3.5)

This rank is $d$, since $C(a) \in K[A]$ for all $a \in K^d$ (so the rank is at most $d$), while $A = C(a)$ for some $a \in K^d$ (so the rank is at least $d$). It follows that
rank}_F M(B) = d \text{ for some } B = C(a) \text{ where } a \in F^d, \text{ i.e., } G = F[B]. \text{ Therefore } K[A] = K[B] \text{ and } A = q(B), \text{ as desired.}

According to Corollary 2.9, the hypotheses of Corollary 3.8 are satisfied if \( A \in M_n(K) \) is diagonalizable and \( A \) commutes with \( A' \). The following example shows that commutativity of \( A \) with \( A' \) is not, by itself, sufficient for the hypotheses of Corollary 3.8 to be satisfied.

**Example 3.9.** Let \( F \) be any field, and let \( K = F(X) \) be the field of rational functions in one variable over \( F \). With the standard derivation, \( K \) is a differential field with field of constants \( F(X^n) \) (where \( p \) is the characteristic of \( F \)). Let \( A \in M_3(K) \) be the matrix

\[
A = \begin{bmatrix}
X^2 & -2X & 1 \\
X^3 & -2X^2 & X \\
X^4 & -2X^3 & X^2
\end{bmatrix}.
\]

Then \( A \) commutes with \( A' \), but \( A \) is not functionally commutative \([A(0) \text{ does not commute with } A(1)]\), so \( A \neq q(B) \) for \( B \in M_3(F) \).

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