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Pao-Liu Chow

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INFINITE-DIMENSIONAL PARABOLIC EQUATIONS IN GAUSS-SOBOLEV SPACES

PAO-LIU CHOW*

Abstract. The paper is concerned with a class of parabolic equations with a gradient-dependent nonlinear term in a Gauss-Sobolev space setting. Under a local Lipschitz continuity condition, it is shown in Theorem 4.2 that there exists a unique strong solution of such a semilinear parabolic equation for which a certain energy inequality holds. The theorem is applied to show the existence of the strong solutions to the Kolmogorov equation and the Hamilton-Jacobi-Bellman equation arising from the control problem for stochastic partial differential equations.

1. Introduction

In finite dimensions, it is well known that the solution of an Itô equation is a diffusion process and the expectation of a smooth function of such a solution satisfies a diffusion equation in $\mathbf{R}^d$, known as the Kolmogorov equation. Therefore it is quite natural to explore such a relationship for the stochastic partial differential equations. The early work on the connection between a diffusion process in a Hilbert space and the infinite-dimensional parabolic and elliptic equations was done by Daleskii [5, 6]. More refined and in-depth studies of such problems based on Gross' theory of the abstract Wiener space [10] were carried out by Kuo [11, 12] and Piech [13], among others. In later years, in the development of the Malliavin calculus, further progress had been made in the area of analysis in Wiener spaces, in particular, the Wiener-Sobolev spaces [14].

Consider the stochastic evolution equation in a Hilbert space H:

$$\begin{aligned} du_t &= Au_t dt + dW_t, \quad t \geq 0, \\ u_0 &= v \in H, \end{aligned} \tag{1.1}$$

where A is an unbounded linear operator in H with a dense domain $\mathcal{D}(A) \subset H$, and W_t is a Wiener process in H with the covariance operator R. Then, formally, the expectation functional Ψ_t of the solution of Eq.(1.1) satisfies the Kolmogorov

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equation:

$$\begin{aligned} \frac{\partial}{\partial t}\Psi_t(v) &= \tfrac{1}{2}Tr\,[RD^2\Psi_t(v)] + (Av, \Psi_t(v)), \quad 0 < t < T, \\ \Psi_0(v) &= \Phi(v), \end{aligned} \tag{1.2}$$

where Tr means the trace, $(\cdot,\cdot)$ is the inner product in H, $D\Psi(v)$ denotes the Fréchet derivative of Ψ at $v \in H$, and Φ is a given function in H. Notice that, since A is unbounded, the term $(Av, D\Phi(v))$ in (1.2) is undefined. As it stands, the equation (1.2) can only be defined for $v \in \mathcal{D}(A)$, which may be a thin set with respect to a reference measure in H. To overcome this difficulty, we proposed to adopt the invariant Gaussian measure μ for the equation (1.1) as the reference measure and to study Eq. (1.2) in a $L^2(H,\mu)$-setting [1]. Based on the measure μ, the equation (1.2) can be defined for almost every (a.e.) $v \in H$. Moreover, by introducing appropriate $L^2(\mu)$-Sobolev spaces, the differential operator $\mathcal{A}\Phi = \frac{1}{2}Tr\,[RD^2\Phi] + (A\cdot, D\Phi)$ in (1.2) acts like the Laplacian operator in finite dimensions. A more comprehensive study of such spaces, called the Gauss-Sobolev spaces, and related elliptic and parabolic equations was given in a later paper [2].

This paper is concerned with the strong solutions of a class of semilinear parabolic equations in a Gauss-Sobolev space setting. It is an extension of a previous work [4] concerning the elliptic case under weaker conditions. Also the method of proof will be different from that of the afore-mentioned paper. The semilinear parabolic equations to be considered arise from the optimal stochastic control of stochastic partial differential equations, known as Hamilton-Jacobi-Bellman equations. A nice exposition of this subject and other references can be found in the book by Da Prato and Zabczyk [8].

The paper is organized as follows. In Section 2, we recall some basic results in the Gauss-Sobolev spaces to be needed in the subsequent sections. Section 3 pertains to the strong solutions of a linear parabolic equation in Gauss-Sobolev spaces. Some a priori estimates for the Green's operator are given by Lemmas 3.1 and 3.1, and the existence and regularity of the solution is proved in Theorem 3.3. The main result of the paper is presented in Section 4 (Theorem 4.2) concerning the existence of the strong solutions to a class of parabolic equations with a gradient-dependent nonlinear terms under a local Lipschitz condition. The proof is based on the basic estimates for the associated Green's operator obtained in Section 3 and the contraction mapping principle. However it is necessary to introduce an equivalent norm to make this principle work. In Section 5, the main Theorem 4.2 is applied to prove the existence of the strong solutions to a Komogorov equation and the Hamilton-Jacobi-Bellman equation for a stochastic control problem.

2. Preliminaries

Let H be a real separable Hilbert space with inner product $(\cdot,\cdot)$ and norm $|\cdot|$ and let $V \subset H$ be a Hilbert subspace with norm $\|\cdot\|$. Denote the dual space of V by V' and their duality pairing by $\langle\cdot,\cdot\rangle$. Assume that the inclusions $V \subset H \cong H' \subset V'$ are dense and continuous.

Suppose that $A : V \to V'$ is a continuous closed linear operator with domain $\mathcal{D}(A)$ dense in H, and W_t is a $H-valued$ Wiener process with the covariance operator R. Consider the linear stochastic equation in a distributional sense:

$$\begin{aligned} du_t &= Au_t\,dt + d\,W_t, \quad t \geq 0, \\ u_0 &= h \in H. \end{aligned} \tag{2.1}$$

Assume that the following conditions (A) hold:

(A.1) Let $A : V \to V'$ be a self-adjoint, coercive operator such that
$$\langle Av, v\rangle \leq -\beta\|v\|^2,$$
for some $\beta > 0$, and $(-A)$ has positive eigenvalues $0 < \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n \leq \cdots$, counting the finite multiplicity, with $\gamma_n \uparrow \infty$ as $n \to \infty$. The corresponding orthonormal set of eigenfunctions $\{e_n\}$ is complete.

(A.2) The resolvent operator $\mathcal{R}_\lambda(A)$ and covariance operator R commute, so that $\mathcal{R}_\lambda(A)R = R\mathcal{R}_\lambda(A)$, where $\mathcal{R}_\lambda(A) = (\lambda I - A)^{-1}$, $\lambda \geq 0$, with I being the identity operator in H.

(A.3) The covariance operator $R : H \to H$ is a self-adjoint operator with a finite trace such that $Tr\,R < \infty$.

Then, by a direct calculation or applying Theorem 4.1 in [3] for invariant measures, we can claim the following lemma.

Theorem 2.1. *Under conditions (A), the stochastic equation (2.1) has a unique invariant measure μ on H, which is a centered Gaussian measure supported in V with covariance operator $\Gamma = -\frac{1}{2}A^{-1}R$.*

Remark: Let e^{tA} denote the semigroup of operators generated by A. Without condition (A.2), the covariance operator of the invariant measure μ is given by $R = \int_0^\infty e^{tA}Re^{tA}dt$, which cannot be evaluated in a closed form. Though a $L^2(\mu)-$theory can be developed in the subsequent analysis, one needs to impose some other conditions which are not easily verifiable. Let $\mathcal{H} = L^2(H, \mu)$ with norm defined by

$$|\!|\!|\Phi|\!|\!| = \{\int_H |\Phi(v)|^2\mu(dv)\}^{1/2},$$

and the inner product $[\cdot,\cdot]$ given by

$$[\Theta, \Phi] = \int_H \Theta(v)\Phi(v)\mu(dv), \quad \text{for}\, \Theta, \Phi \in \mathcal{H}.$$

Let $\mathbf{n} = (n_1, n_2, \cdots, n_k, \cdots)$, where $n_k \in \mathbb{Z}^+$, the set of nonnegative integers, and let $\mathbf{Z} = \{\mathbf{n} : n = |\mathbf{n}| = \sum_{k=1}^\infty n_k < \infty\}$, so that $n_k = 0$ except for a finite number of $n_k's$. Let $h_m(r)$ be the standard one-dimensional Hermite polynomial of degree m. For $v \in H$, define a Hermite (polynomial) functional of degree n by

$$H_{\mathbf{n}}(v) = \prod_{k=1}^\infty h_{n_k}[\ell_k(v)],$$

where we set $\ell_k(v) = (v, \Gamma^{-1/2}e_k)$ and $\Gamma^{-1/2}$ denotes a pseudo-inverse, by restricting it to the range of $\Gamma^{1/2}$. For a smooth functional Φ on H, let $D\Phi$ and $D^2\Phi$ denote the Fréchet derivatives of the first and second orders, respectively. The differential operator

$$\mathcal{A}\Phi(v) = \frac{1}{2}Tr[RD^2\Phi(v)] + \langle Av, D\Phi(v)\rangle \tag{2.2}$$

is well defined for a polynomial functional Φ with $D\Phi(v)$ lies in the domain $\mathcal{D}(A)$ of A. It was shown in [2] that the following holds.

Proposition 2.2. *The set of all Hermite functionals* $\{H_{\mathbf{n}} : \mathbf{n} \in \mathbf{Z}\}$ *forms a complete orthonormal system in* $\mathcal{H}$. *Moreover we have*

$$\mathcal{A}H_{\mathbf{n}}(v) = -\lambda_{\mathbf{n}}H_{\mathbf{n}}(v), \quad \forall\, \mathbf{n} \in \mathbf{Z},$$

where $\lambda_{\mathbf{n}} = \mathbf{n}\cdot\gamma = \sum_{k=1}^{\infty} n_k\gamma_k$.

We now introduce the Gauss-Sobolev spaces. For $\Phi \in \mathcal{H}$, by Proposition 2.2, it can be expressed as

$$\Phi = \sum_{\mathbf{n}\in\mathbf{Z}} \Phi_{\mathbf{n}}H_{\mathbf{n}},$$

where $\Phi_{\mathbf{n}} = [\Phi, H_{\mathbf{n}}]$ and $|||\Phi|||^2 = \sum_{\mathbf{n}} |\Phi_{\mathbf{n}}|^2 < \infty$. Let $\mathcal{H}_m$ denote the Gauss-Sobolev space of order m defined as

$$\mathcal{H}_m = \{\Phi \in \mathcal{H} : |||\Phi|||_m < \infty\}$$

for any integer m, where the norm

$$|||\Phi|||_m = |||(I - \mathcal{A})\Phi)^{m/2}||| = \{\sum_{\mathbf{n}}(1 + \lambda_{\mathbf{n}})^m|\Phi_{\mathbf{n}}|^2\}^{1/2}, \tag{2.3}$$

with I being the identity operator in $\mathcal{H} = \mathcal{H}_0$. For $m \geq 1$, the dual space $\mathcal{H}'_m$ of $\mathcal{H}_m$ is given by $\mathcal{H}_{-m}$, and the duality pairing between them will be denoted by $\langle\langle\cdot,\cdot\rangle\rangle_m$ with $\langle\langle\cdot,\cdot\rangle\rangle_1 = \langle\langle\cdot,\cdot\rangle\rangle$. Clearly, the sequence of norms $\{|||\Phi|||_m\}$ is increasing, that is,

$$|||\Phi|||_m < |||\Phi|||_{m+1},$$

for any integer m, and, by identify $\mathcal{H}$ with its dual $\mathcal{H}'$, we have

$$\mathcal{H}_m \subset \mathcal{H}_{m-1} \subset \cdots \subset \mathcal{H}_1 \subset \mathcal{H} \subset \mathcal{H}_{-1} \subset \cdots \subset \mathcal{H}_{-m+1} \subset \mathcal{H}_{-m}, \quad \text{for}\, m \geq 1,$$

and the inclusions are dense and continuous. Of course the spaces $\mathcal{H}_m$ can be defined for any real number m, but they are needed in this paper.

Owing to the use of the invariant measure μ, as we showed in [1], that it was possible to develop a L^2-theory of infinite-dimensional parabolic and elliptic equations connected to stochastic PDEs similar to the finite-dimensional ones. In particular the following properties of $\mathcal{A}$ are crucial in the subsequent analysis. So far the differential operator $\mathcal{A}$ given by (2.2) is defined only in the linear span of Hermite polynomial functionals. In fact it can be extended to be a self-adjoint linear operator in $\mathcal{H}$. To this end, let $\mathcal{P}_N$ be the projection operator in $\mathcal{H}$ onto its subspace spanned by Hermite polynomial functionals of degree N and define $\mathcal{A}_N = \mathcal{P}_N\mathcal{A}$. Then the following theorem holds (Theorem 3.1, [1]).

Theorem 2.3. *The sequence $\{\mathcal{A}_N\}$ converges strongly to a linear symmetric operator $\mathcal{A} : \mathcal{H}_2 \to \mathcal{H}$, and the following integral identity holds:*

$$\int_H (\mathcal{A}\Phi)\Psi \, d\mu = \int_H (\mathcal{A}\Psi)\Phi \, d\mu = -\frac{1}{2}\int_H (RD\Phi, D\Psi)\, d\mu, \quad \textit{for} \quad \Phi, \Psi \in \mathcal{H}_2.$$

Moreover $\mathcal{A}$ has a self-adjoint extension in $\mathcal{H}$, still denoted by $\mathcal{A}$ with domain $D(\mathcal{A}) = \mathcal{H}_2$.

For $\Phi \in \mathbf{C}_b^2(H)$ being a bounded $\mathbf{C}^2$-continuous functional on H, let P_t denote the transition operator defined by

$$[P_t\Phi](v) = E\{\Phi(u_t)|u_0 = v\} = \Psi_t(v).$$

Then, for $v \in \mathcal{D}(A)$, $\Psi_t(v)$ satisfies the Kolmogorov equation in the classical sense:

$$\begin{aligned} \frac{\partial}{\partial t}\Psi_t(v) &= \mathcal{A}\Psi_t(v), \quad t > 0, \\ \Psi_0(v) &= \Phi(v). \end{aligned} \tag{2.4}$$

In fact the transition operator P_t can be extended to be a bounded linear operator on $\mathcal{H}$ and it is possible to define the equation (2.4) for μ-a.e. $v \in \mathcal{H}$.

Theorem 2.4. *Under conditions (A), the transition operator P_t is defined on $\mathcal{H}$ for all $t \geq 0$ and $\{P_t : t \geq 0\}$ forms a strongly continuous semigroup of linear contraction operators on $\mathcal{H}$ with the infinitesimal generator $\tilde{\mathcal{A}} = \mathcal{A}$ in $\mathcal{H}_2$.*

3. Linear Equation with Basic Estimates

Consider the Cauchy problem for the linear parabolic equation:

$$\begin{aligned} \frac{\partial}{\partial t}\Psi_t(v) &= (\mathcal{A} - \alpha)\,\Psi_t(v) + Q_t(v), \quad \mu - a.e. \; v \in H, \quad 0 < t < T, \\ \Psi_0(v) &= \Phi(v), \end{aligned} \tag{3.1}$$

for $Q \in L^2((0,T);\mathcal{H})$ and $\Phi \in \mathcal{H}$, where the positive parameter α can always be introduced by changing Ψ_t to $e^{\alpha t}\Psi_t$ without effecting the solution behavior in $[0,T]$. We are interested in the solution Ψ of (3.1) in $\mathbf{C}([0,T];\mathcal{H}) \cap L^2((0,T);\mathcal{H}_1)$ and its further regularity properties. To this end, let $\mathcal{G}_t$ denote the Green's operator associated with Equation (3.1) given by

$$\mathcal{G}_t = e^{-\alpha t}P_t. \tag{3.2}$$

In view of Theorem 2.5, the solution of (3.1) can be expressed as

$$\Psi_t(v) = [\mathcal{G}_t\Phi](v) + \int_0^t [\mathcal{G}_{t-s}Q_s](v)ds. \tag{3.3}$$

In what follows, we assume that conditions (A) are satisfied. Then we have the following technical lemmas.

Lemma 3.1. *The Green's operator $\mathcal{G}_t : \mathcal{H} \to \mathcal{H}$ is linear and bounded such that, for $\Phi \in \mathcal{H}$, we have*

$$|||\mathcal{G}_t \Phi||| < |||\Phi|||, \tag{3.4}$$

$$||| \int_0^t \mathcal{G}_s \Phi \, ds|||^2 \leq \{t|||\Phi|||^2\}, \tag{3.5}$$

and, for $\Phi \in \mathcal{H}_{m-1}$ with any integer $m \geq 0$,

$$||| \int_0^t \mathcal{G}_{t-s} \Phi \, ds|||_m^2 \leq \frac{t}{2\alpha_1} |||\Phi|||_{m-1}^2, \quad \text{for} \quad t \in [0, T], \tag{3.6}$$

where $\alpha_1 = (\alpha \wedge 1) = \min\{\alpha, 1\}$, and by convention, we set $||| \cdot |||_0 = ||| \cdot |||$.

Proof. Since P_t is a contraction, clearly the bounds (3.4) and (3.5) hold true. To prove (3.6), we let $\Theta_t = \int_0^t \mathcal{G}_s \Phi \, ds$. By (2.3), we have

$$|||\Theta_t|||_m^2 = \{\sum_{\mathbf{n}} (1 + \lambda_{\mathbf{n}})^m [\Theta_t, H_{\mathbf{n}}]^2\}, \tag{3.7}$$

where

$$\begin{aligned} [\Theta_t, H_{\mathbf{n}}]^2 &= \{\int_0^t [\mathcal{G}_{t-s}\Phi, H_{\mathbf{n}}] \, ds\}^2 \\ &= \{\int_0^t e^{-(\alpha+\lambda_{\mathbf{n}})(t-s)} ds\}^2 [\Phi, H_{\mathbf{n}}]^2 \\ &\leq \frac{t}{2\alpha_1(1+\lambda_{\mathbf{n}})} [\Phi, H_{\mathbf{n}}]^2. \end{aligned} \tag{3.8}$$

Making use of the inequality (3.8), the equation (3.7) yields

$$|||\Theta_t|||_m^2 \leq \frac{t}{2\alpha_1} \{\sum_{\mathbf{n}} (1 + \lambda_{\mathbf{n}})^{m-1} [\Theta_t, H_{\mathbf{n}}]^2\} = \frac{t}{2\alpha_1} |||\Phi|||_{m-1}^2,$$

which verifies (3.6). □

Lemma 3.2. *For $Q \in L^2((0,T); \mathcal{H})$, the following inequality holds:*

$$||| \int_0^t \mathcal{G}_{t-s} Q_s \, ds|||^2 \leq \frac{1}{2\alpha_1} \int_0^t |||Q_s|||^2 \, ds. \tag{3.9}$$

Moreover, for $Q \in L^2((0,T); \mathcal{H}_{m-1})$, we have

$$||| \int_0^t \mathcal{G}_{t-s} Q_s \, ds|||_m^2 \leq \frac{1}{2\alpha_1} \int_0^t |||Q_s|||_{m-1}^2 \, ds. \tag{3.10}$$

Proof. In terms of the Hermite functionals, we can write

$$\mathcal{G}_{t-s} Q_s = \sum_{\mathbf{n}} e^{-\beta_{\mathbf{n}}(t-s)} [Q_s, H_{\mathbf{n}}], \quad \text{with} \quad \beta_{\mathbf{n}} = \alpha + \lambda_{\mathbf{n}}, \tag{3.11}$$

so that

$$||| \int_0^t \mathcal{G}_{t-s} Q_s \, ds|||^2 = \sum_{\mathbf{n}} \{\int_0^t e^{-\beta_{\mathbf{n}}(t-s)} Q_s^{\mathbf{n}} \, ds\}^2, \tag{3.12}$$

where we let $Q_s^{\mathbf{n}} = [Q_s, H_{\mathbf{n}}]$. Since

$$\{\int_0^t e^{-\beta_{\mathbf{n}}(t-s)} Q_s^{\mathbf{n}} \, ds\}^2 \leq \frac{1}{2\beta_{\mathbf{n}}} \int_0^t |Q_s^{\mathbf{n}}|^2 \, ds, \tag{3.13}$$

the equation (3.12) gives the inequality (3.9) as follows

$$|||\int_0^t \mathcal{G}_{t-s}Q_s\,ds|||^2 \le \sum_{\mathbf{n}} \frac{1}{2\beta_{\mathbf{n}}}\int_0^t |Q_s^{\mathbf{n}}|^2\,ds = \frac{1}{2\alpha_1}\int_0^t |||Q_s|||_{-1}^2 ds,$$

where the interchange of the summation and integration can be justified due the monotone convergence.

The inequality (3.10) can be proved similarly. By making use of (3.11) and (3.13), we can get

$$\begin{aligned} |||\int_0^t \mathcal{G}_{t-s}Q_s\,ds|||_m^2 &= \sum_{\mathbf{n}}(1+\lambda_{\mathbf{n}})^m\{\int_0^t e^{-\beta_{\mathbf{n}}(t-s)}Q_s^{\mathbf{n}}\,ds\}^2 \\ &\le \sum_{\mathbf{n}} \frac{(1+\lambda_{\mathbf{n}})^m}{2(\alpha+\lambda_{\mathbf{n}})}\int_0^t |Q_s^{\mathbf{n}}|^2\,ds \\ &\le \frac{1}{2\alpha_1}\int_0^t |||Q_s|||_{m-1}^2\,ds, \end{aligned}$$

as required to be shown. □

In view of the above lemmas, the integral form of solution Ψ_t, to the Cauchy problem (3.1), given by (3.3) belongs to $\mathbf{C}([0,T];\mathcal{H})$ and it is known as a mild solution. In fact it is a strong solution in the sense that, $\Psi \in \mathbf{C}([0,T];\mathcal{H}) \cap L^2((0,T);\mathcal{H}_1)$, and, for any $\Theta \in \mathcal{H}_1$, the following equation holds:

$$[\Psi_t,\Theta] = [\Phi,\Theta] + \int_0^t \langle\langle(\mathcal{A}-\alpha I)\Psi_s,\Theta\rangle\rangle\,ds + \int_0^t [Q_s,\Theta]\,ds, \quad \forall\ t\in[0,T]. \tag{3.14}$$

Theorem 3.3. *Let $\Phi \in \mathcal{H}$ and $Q \in L^2((0,T);\mathcal{H})$. Then Ψ_t given by (3.3) is the strong solution of the Cauchy problem (3.1).*

Proof. From the estimates given in Lemmas 3.1 and 3.2, it follows that $\Psi \in \mathbf{C}([0,T];\mathcal{H}) \cap L^2((0,T);\mathcal{H}_1)$. To show that the equation (3.14) is satisfied, we let $\Psi_t = U_t + V_t$ in (3.3), where

$$U_t = \mathcal{G}_t\Phi, \quad V_t = \int_0^t \mathcal{G}_{t-s}Q_s\,ds.$$

It suffices to show that U and V satisfy, respectively, the following equations:

$$[U_t,\Theta] = [\Phi,\Theta] + \int_0^t \langle\langle(\mathcal{A}-\alpha I)U_s,\Theta\rangle\rangle ds, \tag{3.15}$$

and

$$[V_t,\Theta] = \int_0^t \langle\langle(\mathcal{A}-\alpha I)V_s,\Theta\rangle\rangle ds + \int_0^t [Q_s,\Theta] ds, \quad \forall \quad \Theta\in\mathcal{H}_1. \tag{3.16}$$

Let $\mathcal{P}_N : \mathcal{H} \to \mathcal{H}^N$ be the orthogonal projection from $\mathcal{H}$ into its subspace $\mathcal{H}^N$ spanned by the Hermite polynomials $H_{\mathbf{n}}$ of degree $|\mathbf{n}| \le N$ defined by

$$\Theta_N = \mathcal{P}_N\Theta = \sum_{|\mathbf{n}|\le N} [\Theta, H_{\mathbf{n}}]H_{\mathbf{n}},$$

which converges strongly in $\mathcal{H}$ to Θ. Since $\Theta_N \in \mathcal{D}(\mathcal{A})$, by the semigroup property P_t (Theorem 2.4), it is easy to see that the equation (3.15) holds with Θ replaced by Θ_N. By taking the limit weakly in each term of this equation as $N \to \infty$, we can obtain the equation (3.15) for $\Theta \in \mathcal{H}_1$.

Let $\mathcal{A}_\alpha = (\mathcal{A} - \alpha I)$. By Theorem 2.3, $\mathcal{A}_\alpha : \mathcal{H}_1 \to \mathcal{H}_1$ is self-adjoint and so is $\mathcal{G}_t$ on $\mathcal{H}$. To verify (3.15), we can write

$$\begin{aligned} \langle\langle \mathcal{A}_\alpha V_t, \Theta_N \rangle\rangle &= [V_t, \mathcal{A}_\alpha \Theta_N] \\ &= \int_0^t [Q_s, \mathcal{G}_{t-s} \mathcal{A}_\alpha \Theta_N]\, ds \\ &= -\int_0^t \frac{\partial}{\partial t} [Q_s, \mathcal{G}_{t-s} \Theta_N]\, ds \end{aligned}$$

It follows from (3.15) that, by interchanging the order of integration and noting the self-adjoint property of $\mathcal{G}_t$,

$$\begin{aligned} \int_0^t \langle\langle \mathcal{A}_\alpha V_s, \Theta_N \rangle\rangle ds &= -\int_0^t \int_0^s \frac{\partial}{\partial s} [Q_r, \mathcal{G}_{s-r} \Theta_N]\, dr ds \\ &= -\int_0^t [Q_r - \mathcal{G}_{t-r} Q_r, \Theta_N]\, dr \\ &= -\int_0^t [Q_s, \Theta_N]\, ds + [V_t, \Theta_N], \end{aligned}$$

or

$$[V_t, \Theta_N] = \int_0^t \langle\langle \mathcal{A}_\alpha V_s, \Theta_N \rangle\rangle ds + \int_0^t [Q_s, \Theta_N]\, ds. \tag{3.17}$$

Since $V \in L^2((0,T); \mathcal{H}_1)$ and $\Theta_N \to \Theta$ strongly in $\mathcal{H}$,

$$\lim_{N\to\infty} \int_0^t \langle\langle \mathcal{A}_\alpha V_s, \Theta_N \rangle\rangle\, ds = \int_0^t \langle\langle \mathcal{A}_\alpha V_s, \Theta \rangle\rangle\, ds.$$

Therefore we can pass to a limit weakly term-wise in (3.17) to obtain the equation (3.16). $\square$

4. Semilinear Parabolic Equations

Instead of the linear case (3.1), consider the Cauchy problem for the nonlinear parabolic equation:

$$\begin{aligned} \frac{\partial}{\partial t} \Psi_t(v) &= (\mathcal{A} - \alpha I)\, \Psi_t(v) + \mathcal{B}_t(v, \Psi_t, D\Psi_t) + Q_t(v), \qquad 0 < t < T, \\ \Psi_0(v) &= \Phi(v), \end{aligned} \tag{4.1}$$

where, for each $t \in [0,T)$, Ψ_t is a real-valued function on H, and, under suitable conditions, the nonlinear term $\mathcal{B}_t : V \times \mathbb{R} \times H \to \mathbb{R}$ can be defined for $t \in (0,T)$ and μ-a.e. in H. Similar to the linear problem, we are interested in the strong solution of (4.1) satisfying the following equation:

$$\begin{aligned}[\Psi_t, \Theta] &= [\Phi, \Theta] + \int_0^t \langle\langle(\mathcal{A}-\alpha)\Psi_s, \Theta\rangle\rangle\, ds \\ &+ \int_0^t [\mathcal{B}_s(\cdot, \Psi_s, D\Psi_s), \Theta]\, ds + \int_0^t [Q_s, \Theta]\, ds, \quad \forall\ \Theta \in \mathcal{H}_1.\end{aligned} \tag{4.2}$$

To impose the conditions on $\mathcal{B}$, assume that the following local linear growth and Lipschitz conditions:

(B.1) There exists a positive function ρ_1 in H with $|||\rho_1||| < \infty$ such that, for $v \in V$,

$$|\mathcal{B}_t(v, r, u)|^2 \le \rho_1(v)(1 + |r|^2 + |R^{1/2}u|^2), \quad \forall \quad t \in (0, T), r \in \mathbb{R}, u \in H.$$

(B.2) There exists a positive function ρ_2 in H with $|||\rho_2||| < \infty$ such that, for $v \in V$,

$$|\mathcal{B}_t(v, r, u) - \mathcal{B}_t(v, r', u')|^2 \le \rho_2(v)\{\,|r - r'|^2 + |R^{1/2}(u - u')|^2\},$$

for any $t \in (0, T)$, $v \in V$, $r, r' \in \mathbb{R}$ and $u, u' \in H$.

Remark: Note that, for $i = 1, 2$, the condition $|||\rho_i||| < \infty$ allows $\rho_i(v)$ to grow like a polynomial in $\|v\|$. For instance, if $\rho_i(v) \le C(1 + \|v\|^{2m})$ for some $C > 0$ and for any integer $m \ge 1$. Then, since μ is a centered Gaussian measure with covariance operator $\Gamma = (-\frac{1}{2}A^{-1}R)$, $\|v\|^2 \le (1/\beta)\langle -Av, v\rangle$ by condition (A.1), and $Tr\, R < \infty$ by condition (A.3), we have

$$\int \|v\|^2 \mu(dv) \le \frac{1}{\beta} \int \langle -Av, v\rangle \mu(dv) = \frac{1}{2\beta} Tr\, R < \infty.$$

Therefore, by using standard estimates for Gaussian moments, we obtain

$$|||\rho_i|||^2 \le C \int (1 + \|v\|^{2m})\mu(dv) = C\{1 + \frac{(2m)!}{(2\beta)^m}(Tr\, R)^m\} < \infty$$

as claimed.

Lemma 4.1. *Suppose the conditions (B.1) and (B.2) hold. Then, for any* $\Theta \in \mathcal{H}_1$*, there exists constant* $C_1 > 0$ *such that*

$$|||\mathcal{B}_t(\cdot, \Theta, D\Theta)|||^2 \le C_1(1 + |||\Theta|||_1^2), \tag{4.3}$$

and, for any $t \in (0, T)$, $\Theta, \Phi \in \mathcal{H}_1$*, there is a constant* $C_2 > 0$ *such that*

$$|||\mathcal{B}_t(\cdot, \Theta, D\Theta) - \mathcal{B}_t(\cdot, \Phi, D\Phi)|||^2 \le C_2 |||\Theta - \Phi|||_1^2. \tag{4.4}$$

Proof. By condition (B.1), for $t \in (0, T)$ and $\Theta \in \mathcal{H}_1$, we have

$$\begin{aligned}|||\mathcal{B}_t(\cdot, \Theta, D\Theta)||| &\le |||(\rho_1)^{1/2}(1 + |\Theta|^2 + |R^{1/2}D\Theta|^2)^{1/2}||| \\ &\le |||\rho_1|||^{1/2}\,\{1 + |||\Theta|||^2 + |||R^{1/2}D\Theta|||^2\,\}^{1/2}.\end{aligned}$$

Since $|||\Theta|||_1^2 = |||\Theta|||^2 + |||R^{1/2}D\Theta|||^2$, the above yields (4.3) with $C_1 = |||\rho_1|||$.

Similarly, the condition (B.2) implies that

$$\begin{aligned}&|||\mathcal{B}_t(\cdot,\Theta,D\Theta)-\mathcal{B}_t(\cdot,\Phi,D\Phi)|||\\&=|||(\rho_2)^{1/2}(|\Theta-\Phi|^2+|R^{1/2}D(\Theta-\Phi)|^2)^{1/2}|||\\&\le |||\rho_2|||^{1/2}\{\,|||\Theta-\Phi|||^2+|||R^{1/2}D(\Theta-\Phi)|||^2\,\}^{1/2}.\end{aligned}$$

It follows that (4.4) holds with $C_2=|||\rho_2|||$. □

Theorem 4.2. *Suppose that the conditions (A.1)-(A.3) and (B.1)-(B.2) are satisfied. Then, for* $\Phi\in\mathcal{H}$ *and* $Q\in L^2((0,T);\mathcal{H})$*, the Cauchy problem has a unique strong solution* $\Psi\in\mathbf{C}([0,T];\mathcal{H})\cap L^2((0,T);\mathcal{H}_1)$*. Moreover the following inequality holds:*

$$\sup_{0\le t\le T}|||\Psi_t|||^2+\int_0^T|||\Psi_s|||_1^2\le K(T)\{1+|||\Phi|||^2+\int_0^T|||Q_s|||^2\},\tag{4.5}$$

where $K(T)$ *is a positive constant depending on* T*.*

Proof. The proof is based on the contraction mapping principle in a Banach space. To this end, let $X_T=\mathbf{C}([0,T];\mathcal{H})\cap L^2((0,T);\mathcal{H}_1)$ denote the Banach space of real-valued functions Ψ on H with the norm $|||\cdot|||_T$ defined by

$$|||\Psi|||_T=\{\sup_{0\le t\le T}|||\Psi_t|||^2+\int_0^T|||\Psi_s|||_1^2\}^{1/2}.$$

For $\Psi\in X_T$, consider the linear Cauchy problem:

$$\begin{aligned}\frac{\partial}{\partial t}U_t(v)&=(\mathcal{A}-\alpha)\,U_t(v)+\mathcal{B}(v,\Psi_t,D\Psi_t)+Q_t(v),\quad 0<t<T,\\ \Psi_0(v)&=\Phi(v),\end{aligned}$$

By (4.3) in Lemma 4.1, given $\Psi\in X_T$, $\mathcal{B}(\cdot,\Psi,D\Psi)\in L^2((0,T);\mathcal{H})$. By making use of Theorem 3.3, the problem has a unique strong solution U_t given by

$$U_t=\mathcal{F}_t(\Psi)=\mathcal{G}_t\Phi+\int_0^t\mathcal{G}_{t-s}\mathcal{B}_s(\cdot,\Psi_s,D\Psi_s))ds+\int_0^t\mathcal{G}_{t-s}Q_s ds,\tag{4.6}$$

which shows that the solution operator $\mathcal{F}:X_T\to X_T$ is well defined. To show that $\mathcal{F}$ is a contraction mapping in X_T, let $\Psi,\hat\Psi\in X_T$. First, by invoking (3.9) in Lemma 3.2, (4.4)in Lemma 4.1 and (4.6), we have

$$\begin{aligned}&|||\mathcal{F}_t(\Psi)-\mathcal{F}_t(\hat\Psi)|||^2\\&=|||\int_0^t\mathcal{G}_{t-s}[\mathcal{B}_s(\cdot,\Psi_s,D\Psi_s)-\mathcal{B}_s(\cdot,\hat\Psi_s,D\hat\Psi_s))ds|||^2\\&\le\frac{1}{2\alpha_1}\int_0^t|||\mathcal{B}_s(\cdot,\Psi_s,D\Psi_s)-\mathcal{B}_s(\cdot,\hat\Psi_s,D\hat\Psi_s)|||^2\,ds\\&\le\frac{C_2}{2\alpha_1}\int_0^t|||\Psi_s-\hat\Psi_s|||_1^2\,ds.\end{aligned}\tag{4.7}$$

Next, by making use of (3.10), (4.4) and (4.6), we can deduce that

$$\begin{aligned}
&\int_0^T \|\!|\mathcal{F}_t(\Psi) - \mathcal{F}_t(\hat{\Psi})|\!\|_1^2 dt \\
&\leq \int_0^T \|\!| \int_0^t \mathcal{G}_{t-s}[\mathcal{B}_s(\cdot, \Psi_s, D\Psi_s) - \mathcal{B}_s(\cdot, \hat{\Psi}_s, D\hat{\Psi}_s))ds|\!\|_1^2 dt \\
&\leq \frac{1}{2\alpha_1} \int_0^T \int_0^t \|\!|\mathcal{B}_s(\cdot, \Psi_s, D\Psi_s) - \mathcal{B}_s(\cdot, \hat{\Psi}_s, D\hat{\Psi}_s))|\!\|^2 \, dsdt \\
&\leq \frac{C_2 T}{2\alpha_1} \int_0^T \|\!|\Psi_t - \hat{\Psi}_t|\!\|_1^2 \, dt.
\end{aligned} \tag{4.8}$$

It follows from (4.7) and (4.8) that

$$\|\!|\mathcal{F}_t(\Psi) - \mathcal{F}_t(\tilde{\Psi})|\!\|_T^2 \leq \frac{C_2}{2\alpha_1}(1+T) \int_0^T \|\!|\Psi_t - \hat{\Psi}_t|\!\|_1^2 \, dt,$$

which shows that $\mathcal{F}$ cannot be a contraction map in the norm $\|\!| \cdot |\!\|_T$ no matter how small T is. To circumvent this difficulty, we introduce an equivalent norm $\|\!| \cdot |\!\|_{\lambda,T}$ defined by

$$\|\!|\Psi|\!\|_{\lambda,T} = \{ \sup_{0 \leq t \leq T} \|\!|\Psi_t|\!\|^2 + \lambda \int_0^T \|\!|\Psi_s|\!\|_1^2 \}^{1/2},$$

where $\lambda \geq 1$ is a parameter to be chosen properly. In view of (4.7) and (4.8), we can obtain

$$\begin{aligned}
\|\!|\mathcal{F}_t(\Psi) - \mathcal{F}_t(\tilde{\Psi})|\!\|_{\lambda,T}^2 \quad &\leq \quad \frac{C_2}{2\alpha_1}(1+\lambda T) \int_0^T \|\!|\Psi_t - \hat{\Psi}_t|\!\|_1^2 \, dt \\
&\leq \quad \lambda \frac{C_2}{2\alpha_1}(T + \frac{1}{\lambda}) \int_0^T \|\!|\Psi_t - \hat{\Psi}_t|\!\|_1^2 \, dt.
\end{aligned} \tag{4.9}$$

Let T be so small that $(C_2 T/2\alpha_1) \leq (1/4)$ and, in the meantime, choose λ large enough so that $(C_2/2\alpha_1\lambda) \leq (1/4)$. Then the inequality (4.9) yields

$$\|\!|\mathcal{F}_t(\Psi) - \mathcal{F}_t(\tilde{\Psi})|\!\|_{\lambda,T} \leq \frac{1}{2} \|\!|\Psi - \hat{\Psi}|\!\|_{\lambda,T},$$

which shows that $\mathcal{F}$ is a contraction map in the equivalent norm $\|\!| \cdot |\!\|_{\lambda,T}$. The fixed point $U = \Psi$ is the unique strong solution of the Cauchy problem (4.1) which can be shown to satisfy the variational equation (4.2) similar to the linear case.

To verify the energy inequality (4.5), notice that, from the equation (4.1), it can be seen that $\frac{\partial}{\partial t}\Psi_t \in \mathcal{H}_{-1}$. Thus we have

$$\frac{\partial}{\partial t} \|\!|\Psi_t|\!\|^2 = 2\langle\langle \frac{\partial}{\partial t}\Psi_t, \Psi_t \rangle\rangle,$$

which is integrated to give

$$
\begin{aligned}
&|||\Psi_t|||^2 = |||\Phi|||^2 + 2\int_0^t \langle\langle \frac{\partial}{\partial s}\Psi_s, \Psi_s\rangle\rangle\, ds \\
&= |||\Phi|||^2 + 2\int_0^t \{\langle\langle(\mathcal{A}-\alpha I)\Psi_s, \Psi_s\rangle\rangle + [\mathcal{B}_s(\cdot, \Psi_s, D\Psi_s), \Psi_s] + [Q_s, \Psi_s]\}\, ds \\
&\le |||\Phi|||^2 - 2\int_0^t \{\frac{1}{2}[RD\Psi_s, D\Psi_s] + \alpha|||\Psi_s|||^2\}\, ds \\
&\quad +2\int_0^t \{[\mathcal{B}(\cdot, \Psi_s, D\Psi_s), \Psi_s] + [Q_s, \Psi_s]\}\, ds,
\end{aligned}
$$

where use was made of Equation (4.1) and Theorem 2.3. Hence there is $\alpha_2 > 0$ such that

$$
\begin{aligned}
&|||\Psi_t|||^2 + 2\alpha_2 \int_0^t |||\Psi_s|||_1^2\, ds \\
&\le |||\Phi|||^2 + \varepsilon \int_0^t |||\mathcal{B}_s(\cdot, \Psi_s, D\Psi_s)|||^2\, ds \\
&+(1+\frac{1}{\varepsilon})\int_0^t |||\Psi_s|||^2\, ds + \int_0^t |||Q_s|||^2\, ds,
\end{aligned}
\tag{4.10}
$$

for any $\varepsilon > 0$. By invoking the inequality (4.4) and choosing $\varepsilon = \alpha_2/C_1$, we can deduce from (4.10) that

$$
\begin{aligned}
&|||\Psi_t|||^2 + \alpha_2 \int_0^t |||\Psi_s|||_1^2\, ds \\
&\le |||\Phi|||^2 + \alpha_2 T + C_3 \int_0^t |||\Psi_s|||^2\, ds + \int_0^t |||Q_s|||^2\, ds,
\end{aligned}
\tag{4.11}
$$

for some constant $C_3 > 0$. The above implies that

$$
\sup_{0\le t\le T} |||\Psi_t|||^2 \le |||\Phi|||^2 + \alpha_2 T + C_3 \int_0^T \sup_{0\le s\le t} |||\Psi_s|||^2\, dt + \int_0^T |||Q_s|||^2\, ds,
$$

which, with aid of the Gronwall's inequality, yields

$$
\begin{aligned}
\sup_{0\le t\le T} |||\Psi_t|||^2 \quad &\le \quad e^{C_3 T}(\, |||\Phi|||^2 + \alpha_2 T + \int_0^T |||Q_s|||^2\, ds\,) \\
&\le \quad K_1(T)(1 + |||\Phi|||^2 + + \int_0^T |||Q_s|||^2\, ds),
\end{aligned}
\tag{4.12}
$$

for some constant $K_1(T) > 0$, depending on T. It also follows from (4.11) that

$$
\begin{aligned}
\alpha_2 \int_0^T |||\Psi_t|||_1\, dt \quad &\le \quad |||\Phi|||^2 + \alpha_2 T + \textstyle\int_0^T |||Q_t|||^2\, dt \\
&\quad\quad +C_3 \int_0^T \sup_{0\le s\le t} |||\Psi_s|||^2\, dt.
\end{aligned}
\tag{4.13}
$$

By means of (4.12), we can obtain from (4.13) that

$$
\int_0^T |||\Psi_t|||_1\, dt \le K_2(T)(1 + |||\Phi|||^2 + \int_0^T |||Q_t|||^2\, dt),
\tag{4.14}
$$

for some constant $K_2(T) > 0$. Now the energy inequality (4.5) follows from (4.12) and (4.14). $\square$

5. Applications

5.1. Kolmogorov Equation. Consider the stochastic evolution equation in V':

$$\begin{aligned} du_t &= Au_t dt + F(u_t)dt + dW_t, \quad t \geq 0, \\ u_0 &= v, \end{aligned} \tag{5.1}$$

where $A : V \to V'$, $F : V \to H$, W_t is a Wiener process in H with covariance operator R, and $v \in H$.

We are interested in the strong solution of the Kolmogorov equation associated with Equation (5.1):

$$\begin{aligned} &\frac{\partial}{\partial t} U_t(v) = \mathcal{A}\, U_t(v) + (F(v), DU_t(v)), \quad 0 < t < T, \\ &U_0(v) = \Phi(v), \end{aligned} \tag{5.2}$$

for μ-a.e. $v \in H$, where, as before,

$$\mathcal{A}\Psi(v) = \frac{1}{2} Tr[RD^2\Psi(v)] + \langle Av, D\Psi(v)\rangle.$$

Let $U_t = e^{\alpha t}\Psi_t$ for some $\alpha > 0$. Then Ψ_t satisfies the linear equation:

$$\begin{aligned} \frac{\partial}{\partial t}\Psi_t(v) &= (\mathcal{A} - \alpha I)\,\Psi_t(v) + (F(v), D\Psi_t(v)), \quad 0 < t < T, \\ \Psi_0(v) &= \Phi(v), \end{aligned} \tag{5.3}$$

As mentioned before, instead of (5.2), it suffices to prove that the modified equation (5.3) has a unique strong solution. To this end, let H_0 be the completion of $R^{1/2}H$ in H with respect to the norm $\|\cdot\|_0$ defined by $\|v\|_0 = |R^{-1/2}v|$. In addition to conditions (A.1)–(A.3), we assume that (A.4) $F : V \to H_0$ such that

$$\|F(v)\|_0^2 \leq C(1 + \|v\|^{2m}),$$

for some constant $C > 0$ and for any integer $m \geq 1$.

Theorem 5.1. *Suppose that the conditions (A.1)–(A.4) hold true. Then, for $\Phi \in \mathcal{H}$, the Cauchy problem for the Kolmogorov equation (5.2) has a unique strong solution $\Psi \in \mathbf{C}([0,T];\mathcal{H}) \cap L^2((0,T);\mathcal{H}_1)$. Moreover the following inequality holds:*

$$\sup_{0\leq t\leq T} |||\Psi_t|||^2 + \int_0^T |||\Psi_s|||_1^2\, ds \leq K(T)\{1 + |||\Phi|||^2\},$$

where $K(T)$ is a positive constant depending on T.

Proof. As mentioned before, it suffices to consider equation (5.3), which is a special case of (4.1), where

$$\mathcal{B}_t(v, r, u) = \mathcal{B}(v, u) = (F(v), u), \tag{5.4}$$

is independent of t and r with $Q_t = 0$. To apply Theorem 4.2, we need to show the conditions (B.1) and (B.2) are satisfied. In view of (5.4) and condition (A.4), for $v \in V, u \in H$, we have

$$\begin{aligned} |\mathcal{B}(v,u)|^2 &= |(F(v),u)|^2 \leq \|F(v)\|_0^2 \, |R^{1/2}u|^2 \\ &\leq C(1+\|v\|^{2m})(1+|R^{1/2}u|^2). \end{aligned}$$

Thus condition (B.1) holds with $\rho_1(v) = C(1+\|v\|^{2m})$ and $|\!|\!|\rho_1|\!|\!| < \infty$. Similarly,

$$\begin{aligned} |\mathcal{B}(v,u) - \mathcal{B}(v,u')|^2 &= |(F(v),u-u')|^2 \leq \|F(v)\|_0^2 \, |R^{1/2}(u-u')|^2 \\ &\leq C(1+\|v\|^{2m})(1+|R^{1/2}(u-u')|^2), \end{aligned}$$

which implies condition (B.2). Therefore it follows from Theorem 4.2 that the equation (5.3) has a unique strong solution and so is the Kolmogorov equation (5.2) with the depicted regularity properties. □

Remark: The existence theorem of the strong solution to the Komogorov equation (5.2) was proved in our paper [2] by a different method under a stronger condition than (A.4), that is, $\sup_{v\in H} \|F(v)\|_0 < C$ for μ-a.s..

5.2. Hamilton-Jacobi-Bellman Equation. Consider the optimal control problem with the state equation in a Hilbert space H:

$$\begin{aligned} du_t &= Au_t dt + F(u_t,\nu_t)dt + dW_t, \quad t \geq 0, \\ u_0 &= v, \end{aligned} \tag{5.5}$$

where the nonlinear function $F(\cdot,\nu_t)$ now depends on the control ν_t, a $\mathcal{F}_t$-adapted process with values in a set $\mathcal{K}$. The problem is to find, from the set $\mathcal{K}_T$ of admissible controls ν, the optimal $\nu*$ that minimizes the cost function:

$$J(t,v,\nu) = E\{\int_t^T e^{-\alpha s} B(u_s,\nu_s)\, ds + \Phi(u_T) \,|\, u_t = v\},$$

where $B : H \times \mathcal{K} \to \mathbb{R}^+$ is the running cost function with the discount rate $\alpha > 0$, and $\Phi \in \mathcal{H}$ is the terminal cost. Let V_t denote the optimal cost or the value function given by

$$V_t(v) = \inf_{\nu\in\mathcal{K}} J(t,v,\nu) = J(t,v,\nu*).$$

By applying that the dynamic programming principle, the function $\Psi_t = V_{T-t}(v)$ satisfies the H-J-B (Hamilton-Jacobi-Bellman) equation [9]:

$$\begin{aligned} \frac{\partial}{\partial t}\Psi_t(v) &= (\mathcal{A} - \alpha I)\,\Psi_t(v) + \mathcal{B}(v, D\Psi_t), \qquad 0 < t < T, \\ \Psi_0(v) &= \Phi(v), \end{aligned} \tag{5.6}$$

where

$$\mathcal{B}(v, D\Theta) = \inf_{\nu\in\mathcal{K}} \{(F(v,\nu), D\Theta) + B(v,\nu)\}. \tag{5.7}$$

We assume that the following conditions hold:

(C.1) There exists a positive constant b_1 such that
$$\|F(v,\nu)\|_0^2 = |R^{-1/2}F(v,\nu)|^2 \le b_1(1+\|v\|^{2m}),$$
for any $v \in V$, $\nu \in \mathcal{K}$ and an integer $m \ge 1$.

(C.2) There exists a positive constant b_2 such that $B(\cdot,\cdot) : V \times \mathcal{K} \to \mathcal{R}^+$ has the following bound:
$$|\mathcal{B}(v,\nu)| \le b_2(1+\|v\|^{2m}),$$
for any $v \in V$, $\nu \in \mathcal{K}$ and $m \ge 1$.

Theorem 5.2. *Let the conditions (A.1)–(A.3), (C.1) and (C.2) are satisfied. Then, for $\Phi \in \mathcal{H}$, the H-J-B equation (5.6) has a unique strong solution $\Psi \in \mathbf{C}([0,T];\mathcal{H}) \cap L^2((0,T);\mathcal{H}_1)$. Moreover the following inequality holds:*
$$\sup_{0\le t\le T} \|\!|\Psi_t\|\!|^2 + \int_0^T \|\!|\Psi_s\|\!|_1^2\, ds \le K(T)\{1+\|\!|\Phi\|\!|^2\},$$
where $K(T)$ is a positive constant depending on T.

Proof. To apply Theorem 4.2, we will verify the conditions (B.1) and (B.2). By (5.7) and the assumptions (C.1), (C.2), we can get
$$\begin{aligned} |\mathcal{B}(v,u)|^2 &= |\inf_{\nu\in\mathcal{K}}\{(F(v,\nu),u)+B(v,\nu)\}|^2 \\ &\le 2\{\|F(v,\nu)\|_0^2\,|R^{1/2}u|^2 + |B(v,\nu)|\}^2\} \\ &\le 2b(1+\|v\|^{2m})(1+|R^{1/2}u|^2), \quad v\in V,\ u \in H, \end{aligned} \tag{5.8}$$
where $b = (b_1 \vee b_2)$. Hence the condition (B.1) is met with $\rho_1(v) = 2b(1+\|v\|^{2m})$.

Similarly, by (5.7) and condition (C.1), we have
$$\begin{aligned} |\mathcal{B}(v,u)-\mathcal{B}(v,u')|^2 &= |\inf_{\nu\in\mathcal{K}}\{(F(v,\nu),u)+B(v,\nu)\} \\ &\quad -\inf_{\nu\in\mathcal{K}}\{(F(v,\nu),u')+B(v,\nu)\}|^2 \\ &\le \sup_{\nu\in\mathcal{K}} |(F(v,\nu),u-u')|^2 \\ &\le \sup_{\nu\in\mathcal{K}} \|(F(v,\nu)\|_0^2\,|R^{1/2}(u-u')|^2 \\ &\le \rho_2(v)(1+|R^{1/2}(u-u')|^2), \end{aligned}$$
for $v \in V,\ u,u' \in H$, which verifies condition (B.2) with $\rho_2(v) = b_1(1+\|v\|^{2m})$. Therefore Theorem 4.2 can be applied to draw the desired conclusion. □

Remarks: The stationary problem, when the H-J-B equation (5.6) is elliptic, was treated in our paper [4] in a weighted Gauss-Sobolev space. The infinite-dimensional H-J-B equation was studied earlier by Da Prato [7] in the space of continuous functions and further developed by himself and his collaborators. A comprehensive discussion of this subject and more references can be found in his joint book with Zabczyk [8].

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P. L. Chow: Department of Mathematics, Wayne State University, Detroit, Michigan 48202, USA

E-mail address: plchow@math.wayne.edu