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Nicolò Cangiotti
University of Trento, via Sommarive 14, 38123, Italy, nicolo.cangiotti@unitn.it

Sonia Mazzucchi
University of Trento, via Sommarive 14, 38123, Italy, sonia.mazzucchi@unitn.it

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OGAWA INTEGRABILITY AND A CONDITION FOR CONVERGENCE IN THE MULTIDIMENSIONAL CASE

NICOLÒ CANGIOTTI* AND SONIA MAZZUCCHI

ABSTRACT. The Ogawa stochastic integral is shortly reviewed and formulated in the framework of abstract Wiener spaces. The condition of universal Ogawa integrability in the multidimensional case is investigated by exploiting Ramer's functional, proving that it cannot hold in general without the introduction of a "renormalization term". Explicit examples are provided.

1. Introduction

After the introduction of stochastic integral in the 1940s due to Kiyosi Itô and the developments of the Itô calculus in the succeeding years, particular interest has been devoted to the hypothesis of causality, which are fundamental in stochastic integration theory. In fact, the Itô calculus relies upon concepts as adapted processes, filtration, martingale, conditions that seems to be consistent with a sort of principle of causality in physics. Hence, for many years, the stochastic problems arising in physical modelling (e.g. the phenomenon of diffusion) could be effectively formulated using Itô calculus. Furthermore, the theory of martingales underlying in the Itô calculus provides a powerful tool.

However, at the end of 1960s, the interest to construct a new stochastic theory independently from causality conditions began to take hold. In this context, many approaches have been developed. In particular Anatoliy Skorokhod defined, in 1970s, the so-called Skorokhod integral [22] and introduced the *anticipative calculus*. A few years later, in 1979, Shigeyoshi Ogawa independently introduced the so-called Ogawa integral and the corresponding *noncausal calculus* [16]. In this note, we focus on the latter with the aim to generalize the conditions for Ogawa integrability in the multidimensional case.

There are many approaches to the noncausal stochastic calculus (see e.g. [12]). The Ogawa integral was extensively studied also in relation with the Skorohod integral [13] and the Stratonovich integral [15]. The definition of Ogawa integral has been extended even to the case of random fields [3, 14, 18]; however a detailed study of the case where the integrand function is d-dimensional (with $d \geq 2$) is still lacking. In the present paper we are going to show that in the multidimensional case the condition of universal integrability cannot be fulfilled, even in rather simple cases. For a more recent approach on the generalization of stochastic integral,

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* Corresponding author.

involving not adapted stochastic processes, see the construction due to Wided Ayed and Hui-Hsiung Kuo in [2].

The paper is organized as follows. In section 2 we shortly review the definition and the main results on Ogawa integration. In section 3 we prove the main theorem by applying the theory of abstract Wiener spaces [4, 5]. Section 4 provides some examples.

2. A Short Survey on the Ogawa Integral

In the following we shall adopt Ogawa's recent notation [19, 20]. Let us set a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $(W_t)_{t\in[0,1]}$ be the standard Wiener process with natural filtration $\{\mathcal{F}_t\}$. We define $\mathbf{H}$ as the set of real valued functions $f : [0,1]\times\Omega \to \mathbb{R}$ which are measurable with respect to $B_{[0,1]}\times\mathcal{F}$ and such that the following condition holds:

$$\mathbb{P}\left(\int_0^1 |f(t,\omega)|^2 dt < \infty\right) = 1.$$

Given an orthonormal basis $\{\phi_n\}$ of the Hilbert space $L^2([0,1], dt)$, let us consider the following formal random series

$$S_\phi(f) \equiv \sum_{n=1}^{\infty} (f, \phi_n)(\phi_n, \dot{W}) \tag{2.1}$$

where $(f, \phi_n) = \int_0^1 f(t)\bar{\phi}_n(t)dt$ denotes the inner product in $L^2([0,1], dt)$ and $(\phi_n, \dot{W}) := \int_0^1 \phi_n(t)dW_t$. Now we can define a noncausal stochastic integral, i.e. the Ogawa integral.

Definition 2.1. A function $f \in \mathbf{H}$ is said to be ϕ-integrable (i.e. integrable with respect to the basis $\{\phi_n\}$) if the random series (2.1) converges in probability. In this case this sum is denoted $\int_0^1 f d_\phi W_t$ and it is called the *Ogawa integral of f with respect to the basis* $\{\phi_n\}$. A function integrable with respect to the basis $\{\phi_n\}$ is called ϕ-integrable.

In Def. 2.1 the orthonormal basis $\{\phi_n\}$ plays an important role. The requirement of the independence of the existence as well as of the value of the sum (2.1) from the basis $\{\phi_n\}$ leads naturally to the definition of *universal integrability.*

Definition 2.2. Let $f \in \mathbf{H}$. If f is integrable in the sense of Def. 2.1 with respect to any orthonormal basis and the value of the integral does not depend on the basis, then the function is called *universally integrable* (*u-integrable*).

A different way to characterize the Ogawa integral, which comes directly from the Itô-Nisio theorem [7], is the following. We can consider the sequence of approximated processes as follows

$$W_n^\phi(t) = \sum_{i=1}^{n} \int_0^t \phi_i(s)ds \int_0^1 \phi_i(s)dW_s.$$

According to the Itô-Nisio theorem we have that the sequence $\{W_n^\phi\}$ converges uniformly in $t \in [0,1]$ to W_t with probability 1. Hence, the Ogawa integral can also be defined as the limit of a sequence of Stieltjes integrals. In fact the following holds.

Proposition 2.3. *Let $f \in \mathbf{H}$; then f is ϕ-integrable if and only if the sequence*

$$\int_0^1 f dW_n^\phi(t)$$

of Stieltjes integrals converges in probability. In particular we get

$$\lim_{n\to\infty} \int_0^1 f dW_n^\phi(t) = \int_0^1 f d_\phi W_t.$$

It is important to introduce the definition of regularity of an orthonormal basis.

Definition 2.4. An orthonormal basis $\{\phi_n\}$ in $L^2([0,1],dt)$ is called *regular* if

$$\sup_n \|u_n\|_{L^2} < \infty,$$

where

$$u_n(t) = \sum_{i\le n} \phi_i(t) \int_0^t \phi_i(s)ds.$$

Remark 2.5. Two examples of regular basis are trigonometric functions and Haar functions.

Remark 2.6. The existence of a non-regular basis was proved by Pietro Majer and Maria Elvira Mancino in [10].

Remark 2.7. The results concerning the integrability with respect to regular bases and with respect to any orthonormal basis were studied by Ogawa [17] and then, in the context of Malliavin calculus, by David Nualart and Moshe Zakai [13].

3. A Renormalization Term for Multidimensional Ogawa Integral on Abstract Wiener Spaces

In the following we are going to present an equivalent definition of Ogawa integral with respect to Wiener process in the framework of abstract Wiener spaces [4, 5, 8, 9].

Let $(\mathcal{H}, \langle\, ,\, \rangle)$ be the Hilbert space of absolutely continuous paths $\gamma : [0,1] \to \mathbb{R}^d$ such that $\gamma(0) = 0$ and $\dot\gamma \in L^2([0,1],dt)$ ($\dot\gamma$ denoting the weak derivative of γ), endowed with the inner product

$$\langle \gamma, \eta\rangle = \int_0^1 \dot\gamma(s)\cdot\dot\eta(s)ds, \qquad \gamma,\eta\in\mathcal{H}.$$

Let $\|\ \|$ denote the $\mathcal{H}$-norm, namely $\|\gamma\|^2 = \int_0^1 \dot\gamma(s)\cdot\dot\gamma(s)ds$, $\gamma\in\mathcal{H}$.
Let $C = C([0,1];\mathbb{R}^d)$ be the Banach space of continuous paths $\omega : [0,1] \to \mathbb{R}^d$, endowed with the sup-norm $|\ |$ and let $\mathbb{P}$ be the Wiener measure on the Borel σ-algebra $\mathcal{B}(C)$ of C. Since for $\gamma \in \mathcal{H}$ we have $|\gamma| \le \|\gamma\|$, $\mathcal{H}$ is densely embedded in C. Denoted with C^* the topological dual of C, we have the following chain of dense inclusions:

$$C^* \subset \mathcal{H} \subset C. \tag{3.1}$$

In the following, with an abuse of notation we shall denote $\langle \eta, \omega \rangle$ the dual pairing between two elements $\eta \in C^*$ and $\omega \in C$.

The finitely additive standard Gaussian measure μ defined as

$$\mu(\mathcal{C}_{P,D}) = \int_D \frac{e^{-\frac{\|x\|^2}{2}}}{(2\pi)^{n/2}} dx,$$

on the cylinder sets $\mathcal{C}_{P,D} \subset \mathcal{H}$ of the form

$$\mathcal{C}_{P,D} := \{\gamma \in \mathcal{H} \colon P\gamma \in D\},$$

for some finite dimensional projection operator $P : \mathcal{H} \to \mathcal{H}$ (where $\dim(P\mathcal{H}) = n$) and some Borel set $D \subset \mathcal{H}$, does not extend to a σ-additive measure on the generated σ-algebra [9]. Defined the cylinder sets in C as

$$\tilde{\mathcal{C}}_{\eta_1,\dots,\eta_n;E} := \{\omega \in C \colon (\langle \eta_1, \omega \rangle, \dots, \langle \eta_n, \omega \rangle) \in E\},$$

for some $n \in \mathbb{N}$, $\eta_1, \dots, \eta_n \in C^*$ and E a Borel set of $\mathbb{R}^n$, we have that the intersection $\tilde{\mathcal{C}}_{\eta_1,\dots,\eta_n;E} \cap \mathcal{H}$ is a cylinder set in $\mathcal{H}$. According to the fundamental results by Leonard Gross [4, 5], the finite additive measure $\tilde{\mu}$ on the cylinder sets of C, defined as

$$\tilde{\mu}(\tilde{\mathcal{C}}_{\eta_1,\dots,\eta_n;E}) := \mu(\tilde{\mathcal{C}}_{\eta_1,\dots,\eta_n;E} \cap \mathcal{H})$$

extends to a σ-additive Borel measure on C that coincides with the standard Wiener measure $\mathbb{P}$, in such a way that for any $\gamma \in \mathcal{H}$ such that γ is an element of C^* the following holds

$$\int e^{i\langle \gamma, \omega \rangle} d\mathbb{P}(\omega) = e^{-\frac{1}{2}\|\gamma\|^2}.$$

This allows in particular to define, for any $\eta \in C^*$, a centered Gaussian random variable n_η on $(C, \mathcal{B}(C), \mathbb{P})$ given by $n_\eta(\omega) := \langle \gamma, \omega \rangle$. In particular, for $\eta, \gamma \in C^*$, the following holds

$$\mathbb{E}[n_\eta n_\gamma] = \int_0^1 \dot{\eta}(s) \cdot \dot{\gamma}(s) ds = \langle \eta, \gamma \rangle, \tag{3.2}$$

which shows that the map $n : C^* \to L^2(C, \mathbb{P})$ can be extended, by the density of C^* in $\mathcal{H}$, to an unitary operator $n : \mathcal{H} \to L^2(C, \mathbb{P})$.

It is remarkable that, if $\gamma \in \mathcal{H}$, the Gaussian random variable n_γ can be identified with the Paley-Wiener integral of $\dot{\gamma} \in L^2([0,1])$, i.e. $n_\gamma(\omega) = \int_0^1 \dot{\gamma}(s) dW(s)$.

Given an orthogonal projector $P : \mathcal{H} \to \mathcal{H}$ with finite dimensional range, i.e. of the form $P(\gamma) = \sum_{i=1}^n \langle \gamma, e_i \rangle e_i$, with $\{e_1, \dots, e_n\} \subset \mathcal{H}$ orthonormal vectors generating $P(\mathcal{H})$, it is possible to define a random variable $\tilde{P} : C \to \mathcal{H}$ as $\tilde{P}(\omega) = \sum_{i=1}^n n_{e_i}(\omega) e_i$.

Remark 3.1. More generally, a function $F : \mathcal{H} \to E$ on $\mathcal{H}$ with values in a Banach space E is said to admit a *stochastic extension* $\tilde{F} : C \to E$ if for any sequence $\{P_n\}$ of finite dimensional orthogonal projectors $P_n : \mathcal{H} \to \mathcal{H}$ converging strongly to the identity operator I, the sequence of random variables $\{F \circ \tilde{P}_n\}$ converges in probability to a random variable $\tilde{F}$ on C (and the limit does not depend on the sequence $\{P_n\}$). For further information and examples about abstract Wiener spaces and stochastic extensions see, e.g., [1, 9].

In this framework, the definition of Ogawa integral can be reformulated. Let us consider the $d-$dimensional canonical Wiener process, where $(\Omega, \mathcal{F}) = (C, \mathcal{B}(C))$ and $W_t(\omega) = \omega(t)$, $\omega \in C$. Let $\mathbf{f} : [0,1] \times C \to \mathbb{R}^d$ be a function in $\mathbf{H}$. For any orthonormal basis $\{\phi_n\}$ of $L^2([0,1];\mathbb{R}^d)$ we can construct a corresponding orthonormal basis $\{e_n\}$ of $\mathcal{H}$ as $e_n(s) = \int_0^s \phi_n(u)du$. In fact the map $U : L^2([0,1];\mathbb{R}^d) \to \mathcal{H}$ defined by

$$U(\phi)(s) = \int_0^s \phi(u)du, \qquad \phi \in L^2([0,1];\mathbb{R}^d), \tag{3.3}$$

is unitary with inverse given by $U^{-1}(\gamma) = \dot{\gamma}$, $\gamma \in \mathcal{H}$. The finite dimensional approximations of the formal series (2.1) can be equivalently written as

$$\begin{aligned}
&\sum_{i=1}^n \int_0^1 f(t,\omega)\phi_i(t)dt \int_0^1 \phi_i(t)dW_t \\
&= \sum_{i=1}^n n_{e_i}(\omega) \int_0^1 f(t,\omega)\dot{e}_i(t)dt \\
&= \int_0^1 f(t,\omega) \cdot \dot{\gamma}_n(\omega)(t)dt
\end{aligned} \tag{3.4}$$

where

$$\gamma_n(\omega) := \tilde{P}_n(\omega) = \sum_{i=1}^n e_i n_{e_i}(\omega), \qquad \omega \in C. \tag{3.5}$$

According to this notation, we can say that f is ϕ-integrable if the sequence (3.4) converges in probability. Analogously f is defined to be universally Ogawa integrable if the limit does not depend on the sequence ϕ_n (or, equivalently, on the sequence $\{e_n\}$).

In the following we shall show that in the case $d \geq 2$ the condition of universal integrability is too strong and cannot be fulfilled even in the simplest cases.

Let us consider a C^1 vector field $\boldsymbol{\alpha} : \mathbb{R}^d \to \mathbb{R}^d$ and let $\mathbf{f} : [0,1] \times C \to \mathbb{R}^d$ defined as $\mathbf{f}(t,\omega) := \boldsymbol{\alpha}(\omega(t))$, $t \in [0,1]$. Given an orthonormal basis $\{e_n\}$ of $\mathcal{H}$, let us consider the sequence $\{g_n\}$ of real random variables on $(C, \mathcal{B}(C), \mathbb{P})$ defined as

$$g_n(\omega) := \int_0^1 \boldsymbol{\alpha}(\omega(t)) \cdot \dot{\gamma}_n(\omega)(t)dt, \qquad \omega \in C, \tag{3.6}$$

where γ_n is defined in (3.5). In terms of the function $G : C \to \mathcal{H}$ defined as

$$G(\omega)(t) = \int_0^t \boldsymbol{\alpha}(\omega(s))ds, \qquad \omega \in C,\ t \in [0,1], \tag{3.7}$$

the functions $\{g_n\}$ can be represented by the following inner product

$$g_n(\omega) = \langle G(\omega), \tilde{P}_n(\omega) \rangle. \tag{3.8}$$

For $\omega \in C$, let $DG(\omega)$ denote the Fréchet differential of G evaluated in ω, given by:

$$DG(\omega)(\gamma)_j(t) = \int_0^t \nabla\alpha_j(\omega(s)) \cdot \gamma(s)ds, \tag{3.9}$$

where $\gamma \in \mathcal{H}$, and α_j are the components of $\boldsymbol{\alpha}$, with $j = 1, \dots, d$.

We require now two more hypothesis on $\boldsymbol{\alpha}$ and $\nabla\alpha_j$ that will be necessary hereinafter:

(H1) $\displaystyle\int_0^1 \int_{\mathbb{R}^d} |\boldsymbol{\alpha}(x)|^2 \frac{e^{-\frac{|x|^2}{2t}}}{(2\pi t)^{d/2}} dxdt < +\infty;$

(H2) $\displaystyle\int_0^1 \int_{\mathbb{R}^d} |\nabla\alpha_j(x)|^2 \frac{e^{-\frac{|x|^2}{2t}}}{(2\pi t)^{d/2}} dxdt < \infty, \qquad \forall j = 1, \dots d.$

We can now state the main result.

Theorem 3.2. *For any orthonormal basis $\{e_n\}$ of $\mathcal{H}$, the sequence of renormalized finite dimensional approximations of the Ogawa integral, namely the sequence of real random variables $\{h_n\}$ on $(C, \mathcal{B}(C), \mathbb{P})$ defined as*

$$\begin{aligned} h_n(\omega) &= g_n(\omega) - r_n(\omega) \\ &= \langle G(\omega), \tilde{P}_n(\omega)\rangle - \sum_{i=1}^{n} \langle e_i, DG(\omega)e_i\rangle, \end{aligned} \tag{3.10}$$

converges in $L^2(C, \mathbb{P})$ and the limit is independent on the orthonormal basis $\{e_n\}$.

Remark 3.3. We can look at the limit of the random variables h_n as a *renormalized Ogawa integral*.

The proof relies upon the following lemmas.

Lemma 3.4. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map such that $|f|$ and $\|Jf\|_2$ belong to $L^2(\mathbb{R}^n, \mu)$, with $\|Jf\|_2$ denoting the Hilbert-Schmidt norm of the Jacobian of f and μ is the standard centered Gaussian measure on $\mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} \left(f(x) \cdot x - \mathrm{Tr}(Jf(x))\right)^2 d\mu(x) \le \int_{\mathbb{R}^n} \left(|f(x)|^2 + \|Jf(x)\|_2^2\right)^2 d\mu(x), \tag{3.11}$$

where $\mathrm{Tr}(Jf(x))$ is the trace of the Jacobian of f.

For a detailed proof of Lemma 3.4 see [21], where also the following definition is introduced.

Definition 3.5. A function $G : C \to C$ with $G(C) \subset \mathcal{H}$ is said to be *$\mathcal{H}$-differentiable* if for any $\omega \in C$ the function $G(\omega) : \mathcal{H} \to \mathcal{H}$ defined as $G(\omega)(\gamma) = G(\omega+\gamma)$, $\gamma \in \mathcal{H}$, is Fréchet differentiable at the origin in $\mathcal{H}$. Its Fréchet derivative, namely the linear operator $DG(\omega(0)) \in L(\mathcal{H};\mathcal{H})$, will be denoted with the symbol $DG(\omega)$ and called the *$\mathcal{H}$-derivative of G at ω.*

Lemma 3.6 (Ramer's formula)**.** *Let $G : C \to C$, with $G(C) \subset \mathcal{H}$, be a $\mathcal{H}$-differentiable map such that for any $\omega \in C$ the $\mathcal{H}$-derivative $DG(\omega) \in L(\mathcal{H}, \mathcal{H})$ is a Hilbert-Schmidt operator. Let us assume furthermore that the maps $\|G\| : C \to \mathbb{R}$ and $\|DG\|_2 : C \to \mathbb{R}$, where $\|DG(\omega)\|_2$ denotes the Hilbert-Schmidt norm of $DG(\omega)$, belong to $L^2(\Omega, \mathbb{P})$. Let $\{e_i\}$ be an orthonormal basis of $\mathcal{H}$ and let $\{P_n\}$ and $\{\tilde{P}_n\}$ be the sequence of finite dimensional projectors on the span of $e_1, \dots, e_n$ and their stochastic extensions respectively. Then the sequence of random variables $\{h_n\}$ defined as*

$$h_n(\omega) := \langle G(\omega), \tilde{P}_n(\omega)\rangle - \mathrm{Tr}(P_n DG(\omega)), \qquad \omega \in C,$$

converges in $L^2(C,\mathbb{P})$ *and the limit does not depend on the basis* $\{e_i\}$*.*

The proof of Lemma 3.6 is a direct consequence of Lemmas 4.2 and 4.3 in [21].

Proof [of Theorem 3.2]. It is straightforward to verify that the map $G : C \to C$ defined by (3.7) is $\mathcal{H}$-differentiable and its $\mathcal{H}$-derivative DG is given by (3.9). Furthermore, for any $\omega \in C$, the operator $DG(\omega)$ is Hilbert-Schmidt. Indeed $DG(\omega) : \mathcal{H} \to \mathcal{H}$ is unitary equivalent to the linear operator $T : L^2([0,1];\mathbb{R}^d) \to L^2([0,1];\mathbb{R}^d)$ defined as

$$T = U^{-1} \circ DG(\omega) \circ U, \tag{3.12}$$

where $U : L^2([0,1];\mathbb{R}^d) \to \mathcal{H}$ is the unitary operator defined in (3.3). By direct computation it is simple to see that T is explicitly given in terms of a kernel $K \in L^2([0,1]\times[0,1])$, i.e. for $\phi \in L^2([0,1];\mathbb{R}^d)$ and $t \in [0,1]$,

$$(T\phi)_j(t) = \int_0^1 K_j(t,t')\cdot\phi(t')dt', \quad j = 1,\dots,d, \tag{3.13}$$

where $K_j(t,t') = \nabla\alpha_j(\omega(t))\chi_{[0,t]}(t')$, $t,t' \in [0,1]$. By formula 4.32 in [11], the Hilbert-Schmidt norm of T is equal to:

$$\begin{aligned}\|T\|_2^2 &= \int_{[0,1]\times[0,1]} |K(t,t')|^2 dtdt' = \sum_{j=1}^d \int_0^1\int_0^1 |\nabla\alpha_j(\omega(t))|^2\chi_{[0,t]}(t')dtdt' \\ &= \sum_{j=1}^d \int_0^1 t|\nabla\alpha_j(\omega(t))|^2 dt \le \sum_{j=1}^d \int_0^1 |\nabla\alpha_j(\omega(t))|^2 dt < \infty,\end{aligned}$$

where the boundedness of the last expression follows by the continuity of the maps $t \mapsto \nabla\alpha_j(\omega(t))$. By the unitary equivalence of T and $DG(\omega)$, we get

$$\|DG(\omega)\|_2^2 = \sum_{j=1}^d \int_0^1 t|\nabla\alpha_j(\omega(t))|^2 dt < \infty.$$

Moreover, by the hypothesis (H1) and (H2), we have that

$$\begin{aligned}\mathbb{E}[\|G\|^2] &= \int_0^1 \mathbb{E}[|\boldsymbol{\alpha}(\omega(t))|^2]dt \\ &= \int_0^1\int_{\mathbb{R}^d} |\boldsymbol{\alpha}(x)|^2 \frac{e^{-\frac{|x|^2}{2t}}}{(2\pi t)^{d/2}} dxdt < \infty \\ \mathbb{E}[\|DG\|_2^2] &\le \sum_{j=1}^d \int_0^1 \mathbb{E}[|\nabla\alpha_j(\omega(t))|^2]dt \\ &= \sum_{j=1}^d \int_0^1\int_{\mathbb{R}^d} |\nabla\alpha_j(x)|^2 \frac{e^{-\frac{|x|^2}{2t}}}{(2\pi t)^{d/2}} dxdt < \infty.\end{aligned}$$

By Lemma 3.6 the sequence of random variables $\{h_n\}$ given by

$$h_n(\omega) = \langle G(\omega), \tilde{P}_n(\omega)\rangle - \mathrm{Tr}(P_n DG(\omega))$$

converges in $L^2(C,\mathbb{P})$ an the limit does not depend on the orthonormal basis $\{e_i\}$. Furthermore, by direct computation, the "renormalization term" $\mathrm{Tr}(P_n DG(\omega))$ is given by

$$\mathrm{Tr}(P_n DG(\omega)) = \sum_{i=1}^{n} \langle e_i, DG(\omega)e_i\rangle = \sum_{i=1}^{n} \int_0^1 \dot{e}_i(t)\cdot(e_i(t)\cdot\nabla)\boldsymbol{\alpha}(\omega(t))dt.$$

□

Corollary 3.7. *For any orthonormal basis $\{e_n\}$ of $\mathcal{H}$, the sequence h_n defined in Theorem 3.2 converges in probability and the limit is independent of the basis $\{e_n\}$.*

4. Examples

According to Theorem 3.2, the condition of existence of the limit in probability of the sequence of random variables $\{g_n\}$ defined in (3.6), i.e. the Ogawa integrability of the function $f \in \mathbf{H}$, with $f(t,\omega) := \boldsymbol{\alpha}(\omega(t))$, $t \in [0,1]$, with respect to the orthonormal basis $\{\phi_n\}$ of $L^2([0,1];\mathbb{R}^d)$ (with $\phi_n = \dot{e}_n$) is equivalent to the existence of the limit in probability of the "renormalization term" $r_n(\omega) = \mathrm{Tr}(P_n DG(\omega))$. Analogously, the universal Ogawa integrability of f is equivalent to the convergence in probability of r_n to a limit which does not depend on the basis $\{e_n\}$ of $\mathcal{H}$. In particular, if the linear operator $DG(\omega) \in L(\mathcal{H},\mathcal{H})$ is not trace class, then the convergent of sequence $\mathrm{Tr}(P_n DG(\omega))$ is not guaranteed and, in general, its value depends on the orthonormal basis $\{e_n\}$. We are going to show that this problem occurs even in very simple cases.

Let $d = 2$ and $\boldsymbol{\alpha} : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear vector field of the form

$$\boldsymbol{\alpha}(x,y) = (h_1x + k_1y, h_2x + k_2y). \tag{4.1}$$

In this case the map $G : C \to \mathcal{H}$ is given by

$$G(\omega)(t) = \left(h_1\int_0^t \omega_1(s)ds + k_1\int_0^t \omega_2(s)ds, h_2\int_0^t \omega_1(s)ds + k_2\int_0^t \omega_1(s)ds\right),$$

where $\omega = (\omega_1,\omega_2) \in C$. The $\mathcal{H}$-derivative $DG(\omega)$ for any $\omega \in C$ is the linear operator $DG : \mathcal{H} \to \mathcal{H}$ simply given by

$$DG(\gamma)(t) = \left(h_1\int_0^t \gamma_1(s)ds + k_1\int_0^t \gamma_2(s)ds, h_2\int_0^t \gamma_1(s)ds + k_2\int_0^t \gamma_2(s)ds\right),$$

with $\gamma = (\gamma_1,\gamma_2) \in \mathcal{H}$.

We can compute explicitly the spectrum of the self-adjoint operator $|DG| = \sqrt{DG^*DG}$. Indeed, setting for notational simplicity $L \equiv DG^*DG$ we have, for $\eta,\gamma \in \mathcal{H}$:

$$\begin{aligned}\langle \eta, L\gamma\rangle &= \langle DG\eta, DG\gamma\rangle \\ &= \int_0^1 (\eta_1(t),\eta_2(t))A(\gamma_1(t),\gamma_2(t))^T dt,\end{aligned}$$

with

$$A = \begin{pmatrix} h_1^2 + h_2^2 & h_1k_1 + h_2k_2 \\ h_1k_1 + h_2k_2 & k_1^2 + k_2^2 \end{pmatrix}.$$

Hence, for $\gamma \in \mathcal{H}$ the vector $L(\gamma) \in \mathcal{H}$ is given by

$$L(\gamma)(t)^T = -\int_0^t \int_1^s A\gamma(r)^T drds.$$

L is a compact operator and has a discrete spectrum. By introducing in $\mathbb{R}^2$ an orthonormal basis $\{u_1, u_2\}$ of eigenvectors of the symmetric matrix A, with corresponding eigenvalues $a_1, a_2 \in \mathbb{R}^+$, the eigenvectors $\{\gamma_n\}$ of L can be represented as linear combination of u_1 and u_2, namely $\gamma_n = \eta_{n,1}u_1 + \eta_{n,2}u_2$, with $\eta_{n,j} : [0,1] \to \mathbb{R}$. The components $\{\eta_{n,j}\}$ of the eigenvectors (with eigenvalues λ) are solutions of

$$\begin{cases} \lambda_{n,j}\ddot{\eta}_{n,j} + a_j\eta_{n,j} = 0 \\ \dot{\eta}_{n,j}(1) = 0 \\ \eta_{n,j}(0) = 0 \end{cases} \qquad j = 1, 2,$$

which yields in the non-trivial case where $a_j > 0$ the solutions $\lambda_{n,j} = \frac{4a_j}{\pi^2(1+2n)^2}$, with corresponding eigenvectors $\gamma_{n,j}(t) = \sin\left(\left(\frac{\pi}{2} + n\pi\right)t\right)u_j$, where $j = 1, 2$. Hence, we can conclude that $|DG| = \sqrt{L}$ is not trace class and in general the limit of $r_n = \mathrm{Tr}(P_nDG)$ does not necessary exist and, if it exists, its value depends on the sequence of projectors $\{P_n\}$ or, equivalently, on the choice of the orthonormal basis $\{e_n\}$ of $\mathcal{H}$.

It is interesting to investigate the value that the "renormalization term" assumes for different choices of the orthonormal basis $\{e_n\}$, in order to understand the role it plays in a few particular cases.

Let us consider $L^2([0,1];\mathbb{R}^2)$ and the following orthonormal basis

$$\begin{aligned} \{\psi_n\} : &= \Big\{(1,0), (0,1), \sqrt{2}(\cos(2\pi nt), 0), \sqrt{2}(\sin(2\pi nt), 0), \\ &\qquad \sqrt{2}(0, \cos(2\pi nt)), \sqrt{2}(0, \sin(2\pi nt))\Big\} \\ &= \{\psi_{0,x}, \psi_{0,y}, \psi_{n,1}, \psi_{n,2}, \psi_{n,3}, \psi_{n,4}\}, \end{aligned}$$

with $n \in \mathbb{N} \setminus \{0\}$. Rewriting formula (3.13) explicitly, we can compute

$$\langle \psi_n, T\psi_n \rangle = \int_0^1 \psi_n(t) \cdot \left(\int_0^t \psi_n(s)ds \cdot \nabla\right) \boldsymbol{\alpha}(\omega(t))dt,$$

where $\boldsymbol{\alpha} : \mathbb{R}^2 \to \mathbb{R}^2$ is given by (4.1) and $T : L^2([0,1];\mathbb{R}^2) \to L^2([0,1];\mathbb{R}^2)$ is defined by (3.12). For the vectors of the form $\psi_{n,j}$ with $j = 1, \ldots, 4$ we have:

$$\langle \psi_{n,j}, T\psi_{n,j} \rangle = 0;$$

while for the two constant vectors

$$\langle \psi_{0,x}, T\psi_{0,x} \rangle = \frac{h_1}{2};$$
$$\langle \psi_{0,y}, T\psi_{0,y} \rangle = \frac{k_2}{2}.$$

This gives for the basis $\{\psi_n\}$ the following "renormalization term" (depending on the divergence of $\boldsymbol{\alpha}$):

$$r_n = \mathrm{Tr}(P_nDG) = \sum_{i=1}^n \langle \psi_i, T\psi_i \rangle = \frac{1}{2}\nabla \cdot \boldsymbol{\alpha}.$$

Let us now consider a different basis in $L^2([0,1];\mathbb{R}^2)$:

$$\begin{aligned}\{\xi_n\} :&= \{(1,0),(0,1),(\cos(2\pi nt),\sin(2\pi nt)),(\sin(2\pi nt),\cos(2\pi nt)),\\ &\qquad(-\cos(2\pi nt),\sin(2\pi nt)),(-\sin(2\pi nt),\cos(2\pi nt))\}\\ &= \{\xi_{0,x},\xi_{0,y},\xi_{n,1},\xi_{n,2},\xi_{n,3},\xi_{n,4}\},\end{aligned}$$

with $n \in \mathbb{N}\setminus\{0\}$.

We use the same argument as before for the vectors

$$\xi_{n,1} = (\cos(2\pi nt),\sin(2\pi nt)).$$

We obtain:

$$\begin{aligned}\langle \xi_{n,1}, T\xi_{n,1}\rangle &= \int_0^1 \left(k_1\frac{\sin^2(\pi nt)\cos(2\pi nt)}{\pi n} + h_1\frac{\sin(2\pi nt)\cos(2\pi nt)}{2\pi n}\right.\\ &\qquad \left. + k_2\frac{\sin(2\pi nt)\sin^2(\pi nt)}{\pi n} + h_2\frac{\sin^2(2\pi nt)}{2\pi n}\right) dt\\ &= \frac{h_2-k_1}{4n\pi} = \frac{\nabla\times\boldsymbol{\alpha}}{4n\pi}.\end{aligned}$$

Analogously

$$\begin{aligned}\langle \xi_{n,2}, T\xi_{n,2}\rangle &= \int_0^1 \left(h_2\frac{\sin^2(\pi nt)\cos(2\pi nt)}{\pi n} + k_2\frac{\sin(2\pi nt)\cos(2\pi nt)}{2\pi n}\right.\\ &\qquad \left. +h_1\frac{\sin(2\pi nt)\sin^2(\pi nt)}{\pi n} + k_1\frac{\sin^2(2\pi nt)}{2\pi n}\right) dt\\ &= \frac{k_1-h_2}{4n\pi} = -\frac{\nabla\times\boldsymbol{\alpha}(\omega(t))}{4n\pi},\end{aligned}$$

and

$$\begin{aligned}\langle \xi_{n,3}, T\xi_{n,3}\rangle &= \frac{k_1-h_2}{4\pi n} = -\frac{\nabla\times\boldsymbol{\alpha}}{4\pi n}\\ \langle \xi_{n,4}, T\xi_{n,4}\rangle &= \frac{-k_1+h_2}{4\pi n} = \frac{\nabla\times\boldsymbol{\alpha}}{4\pi n}\end{aligned}$$

In this case the series $\sum_{i=1}^n\langle\xi_i,T\xi_i\rangle$ cannot converge absolutely and and the value of the "renormalization term" depends on the order of the terms in the sum.

At last we consider in the Hilbert space $\mathcal{H}$ the sequence of orthogonal projection operators onto the finite dimensional subspaces H_n of piecewise linear paths of the form

$$\gamma(t) = \sum_{i=0}^{n-1}\mathbb{1}_{\left[\frac{i}{n},\frac{i+1}{n}\right]}(t)\left(\gamma(i/n)+n\left(\gamma(i+1/n)-\gamma(i/n)\right)(t-i/n)\right), \tag{4.2}$$

with $t\in[0,1]$. An orthonormal basis of H_n is provided, e.g., by the vectors

$$\{(z_{n,i},0),(0,z_{n,i})\}_{i=0,\dots,n-1}, \tag{4.3}$$

where

$$z_{n,i}(t) = \sqrt{n}\mathbb{1}_{\left[\frac{i}{n},\frac{i+1}{n}\right]}(t)\left(t-\frac{i}{n}\right)+\frac{1}{\sqrt{n}}\mathbb{1}_{\left[\frac{i+1}{n},1\right]}(t),$$

with $i = 0, \dots, n-1$. We also notice that:

$$\dot{z}_{n,i}(t) = \sqrt{n}\mathbb{1}_{\left[\frac{i}{n},\frac{i+1}{n}\right]}(t).$$

It is not difficult to compute

$$\langle (z_{n,i}, 0), DG(z_{n,i}, 0)\rangle = \frac{h_1}{2n}, \quad \langle (0, z_{n,i}), DG(0, z_{n,i})\rangle = \frac{k_2}{2n},$$

Thereby we get

$$\lim_{n\to\infty} \mathrm{Tr}(P_n DG) = \frac{1}{2}\nabla\cdot\boldsymbol{\alpha}. \tag{4.4}$$

This last example is particularly interesting since in the case where $\{P_n\}$ are the projectors on the subspaces of piecewise linear path described above, the limits of the sequences $\{g_n\}$ and $\{r_n\}$ (defined respectively by (3.8) and (3.10)) can be computed explicitly. This provides a possible technique for the computation of the limit of the sequence $\{h_n\}$ for linear vector fields $\boldsymbol{\alpha}$ and, by Theorem (3.2), this limit is independent on the sequence of projectors. We remark that this toy model can be studied also by applying different techniques, such as, for instance, Malliavin calculus [13].

The following lemma provides a useful tool in the proof of theorem 4.2, which shows that Ogawa integral with respect to the basis (4.3) coincides with Stratonovich Integral.

Lemma 4.1. *Let $G : C \to \mathcal{H}$ be a linear operator such that its restriction $G_{\mathcal{H}}$ on $\mathcal{H}$ is Hilbert-Schmidt. Let $\{P_n\}$ be a sequence of finite dimensional projection operators in $\mathcal{H}$ converging strongly to the identity. Then the sequences of random variables $\{g_n\}$ and $\{g'_n\}$ defined as:*

$$\begin{aligned} g_n(\omega) &= \langle G(\omega), \tilde{P}_n(\omega)\rangle, \qquad \omega\in C \\ g'_n(\omega) &= \langle G(\tilde{P}_n(\omega)), \tilde{P}_n(\omega)\rangle, \qquad \omega \in C \end{aligned}$$

satisfy

$$\lim_{n\to\infty} \mathbb{E}[|g_n - g'_n|^2] = 0. \tag{4.5}$$

Proof.

$$\begin{aligned} \mathbb{E}[|g_n - g'_n|^2] &= \int |\langle G(\omega) - G(\tilde{P}_n(\omega)), \tilde{P}_n(\omega)\rangle|^2 d\mathbb{P}(\omega) \\ &= \int |\langle G(\omega - \tilde{P}_n(\omega)), \tilde{P}_n(\omega)\rangle|^2 d\mathbb{P}(\omega) \\ &= \int |\langle G(\sum_{j=n+1}^{\infty} e_j n_{e_j}(\omega)), \sum_{i=1}^{n} e_i n_{e_i}(\omega)\rangle|^2 d\mathbb{P}(\omega) \end{aligned}$$

$$= \sum_{j,j'=n+1}^{\infty} \sum_{i,i'=1}^{n} \langle Ge_j, e_i\rangle \langle Ge_{j'}, e_{i'}\rangle \mathbb{E}[n_{e_j} n_{e_{j'}} n_{e_i} n_{e_{i'}}]$$
$$= \sum_{j=n+1}^{\infty} \sum_{i=1}^{n} (\langle Ge_j, e_i\rangle)^2$$
$$= \sum_{j=n+1}^{\infty} \langle P_n G e_j, P_n G e_j\rangle,$$

where in the third step we have applied Itô-Nisio theorem. By using the assumption that $G_{\mathcal{H}}$ is a Hilbert-Schmidt operator we obtain (4.5). □

Theorem 4.2. *Let $\boldsymbol{\alpha}$ be the linear vector field given by* (4.1) *and $G : C \to \mathcal{H}$ the linear operator* (3.7)*. Then the sequence of random variables $\{g_n\}$ defined by*

$$g_n(\omega) = \langle G(\omega), \tilde{P}_n(\omega)\rangle, \qquad \omega \in C,$$

where $\{P_n\}$ is the sequence of orthogonal projectors onto the subspaces H_n of piecewise linear paths (4.2)*, converges in $L^2(C, \mathbb{P})$ to the Stratonovich integral*

$$\int_0^1 \boldsymbol{\alpha}(\omega(t)) \circ d\omega(t).$$

Proof. By lemma 4.1 the sequence $\{g_n\}$ has the same limit of the sequence $\{g'_n\}$, where

$$g'_n(\omega) = \langle G(\tilde{P}_n(\omega)), \tilde{P}_n(\omega)\rangle, \qquad \omega \in C$$

if such a limit exists. Moreover the random variables $\{g'_n\}$ assume the following form

$$g'_n(\omega) = \int_0^1 \boldsymbol{\alpha}(\omega_n(t)) \cdot \dot{\omega}_n(t) dt,$$

where $\omega_n = \tilde{P}_n \omega \in \mathcal{H}$. By Wong-Zakai approximations results [6], in the case where $\{P_n\}$ are projectors on piecewise linear paths, the sequence $\{g'_n\}$ converges in $L^2(C, \mathbb{P})$ to the Stratonovich integral $\int_0^1 \boldsymbol{\alpha}(\omega(t)) \circ d\omega(t)$. □

Theorem 4.3. *Let $\boldsymbol{\alpha}$ be the linear vector field given by* (4.1) *and $G : C \to \mathcal{H}$ the linear operator* (3.7)*. Then the sequence of random variables $\{h_n\}$ defined in Theorem 3.2, namely*

$$h_n(\omega) = g_n(\omega) - r_n,$$

with $r_n = \mathrm{Tr}(P_n DG)$, converges to the Itô integral.

$$\int_0^1 \boldsymbol{\alpha}(\omega(t)) d\omega(t)$$

and the limit does not depend on the sequence $\{P_n\}$.

Proof. By Theorem 3.2 the sequence $\{h_n\}$ converges in $L^2(C, \mathbb{P})$ and the limit is independent of $\{P_n\}$. In the case where $\{P_n\}$ are projectors onto subspaces of piecewise linear paths, we can compute explicitly the limit of both $\{g_n\}$ and $\{r_n\}$. Indeed, by Theorem 4.2 and formula (4.4), we obtain

$$\begin{aligned}\lim_{n\to\infty} h_n(\omega) &= \lim_{n\to\infty} g_n(\omega) - \lim_{n\to\infty} r_n \\ &= \int_0^1 \boldsymbol{\alpha}(\omega(t)) \circ d\omega(t) - \frac{1}{2}\nabla\cdot\boldsymbol{\alpha}\end{aligned}$$

where the limits are meant in $L^2(C,\mathbb{P})$. By the conversion formula between Itô and Stratonovich integral

$$\int_0^1 \boldsymbol{\alpha}(\omega(t)) \circ d\omega(t) = \int_0^1 \boldsymbol{\alpha}(\omega(t)) d\omega(t) + \frac{1}{2}\int_0^1 \nabla\cdot\boldsymbol{\alpha}(\omega(t)) dt, \qquad (4.6)$$

we obtain the final result. □

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Nicolò Cangiotti: Department of Mathematics, University of Trento, via Sommarive 14, 38123, Italy

Email address: nicolo.cangiotti@unitn.it

Sonia Mazzucchi: Department of Mathematics, University of Trento, via Sommarive 14, 38123, Italy

Email address: sonia.mazzucchi@unitn.it