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SCS 1: More Notes on Spread

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MORE NOTES ON SPREAD

J. LAWSON

Notation and references consistent with "NOTES ON NOTES BY JDL". That work is referenced as H-SP (Hofmann on "Spread"). JDL should perhaps now become JDL I

LEMMA 1. Let \bar{X} be an order generating set in a \mathcal{CL} -object T . Then $IRR T \subset \bar{X}$

Proof. Let $y \in IRR T$, $Y = \bar{X} \wedge \uparrow y$. By hypothesis $y = \inf Y$. Since y is prime in the \mathcal{CL} -object $\uparrow y$, by "THE LEMMA", $y \in \bar{Y} \subset \bar{X}$.

THEOREM

PROP. 2. Let \bar{X} be a subset of a \mathcal{CL} -object T .

TAE

- (1) \bar{X} is generating
- (2) \bar{X} is order-generating
- (3) $IRR T \subset \bar{X}$

Proof. (1) \Leftrightarrow (2) by Thm. 1.10 [H-SP], and (3) \Rightarrow (2) by Prop. 1.4 [H-SP]; (2) \Rightarrow (3) by Lemma 1.

COROLLARY 3. $IRR T$ is the unique smallest closed (order) generating set if $T \in \mathcal{CL}$

A set \bar{X} (in a semilattice S) is algebraically generating if S is the smallest subsemilattice containing \bar{X} .

Prop. 2 and Corollary 3 really wrap up the question of generation and order generation in \mathcal{CL} -objects. Also the ASSERTION on the next page settles my inquiry on p. 5 of (H-SP). There remains the question: $IRR T = IRR \uparrow$ for $T \in \mathcal{Z}$? Be sure to understand thm. 1.12, p. 4 of (H-SP) is superseded by Ref. 3

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PROP. 4, Let $T \in \mathcal{CL}$ and let $\bar{X} = \bigcup_{i=1}^n T_i$

where each T_i is a subsemilattice containing 1 of T

and $n < \infty$. TAE

(1) \bar{X} is generating

(2) \bar{X} is algebraically generating

(3) \exists a continuous epimorphism $m: \prod_{i=1}^n \bar{T}_i \rightarrow T$
defined from the multiplication mapping.

Proof, (3) \Rightarrow (2) Immediate since $\bar{X} = \bar{T}_1 \cup \dots \cup \bar{T}_n$

(2) \Rightarrow (1) Alg. gen. \Rightarrow Ord. gen. ~~Now apply Thm. 1.10 [H-CP]~~

(1) \Rightarrow (3) The image of m is a compact subsemilattice containing \bar{X} . Hence the image is T .

ASSERTION, In [Lawson, Lattices with no interval homomorphisms] Example 1 has IRR T order generating (hence IRR T order generating $\nRightarrow T \in \mathcal{CL}$), In Example 3

IRR T is not order generating. In fact it can be argued that this example contains a compact sublattice L for which $\text{IRR } L = \{1\}$. Hence Prop. 1.4 [H-SP] does not extend to compact semilattices.

We adopt Def. 2.8 of [H-CP] for the "spread" of a \mathcal{CL} -object. If $T \in \mathcal{CL}$, our formulation for spread is that $SP(T)$ is the least cardinal α for which there exists a chain in T whose union is a generating set.

THEOREM 5 Let $T \in \mathcal{L}$ and let n be a positive integer. TAE.

- (1) $SP(T) \leq n$;
- (2) T is generated by a union of n chains;
- (3) T is order-generated by a union of n closed chains;
- (4) T is algebraically generated by a union of n closed chains;
- (5) $\overline{IRR T} \subset \bigcup_{i=1}^n C_i$ where each C_i is a compact chain;
- (6) $IRR T$ has width $\leq n$ (Width \equiv largest cardinality of an anti-chain)
- (7) \exists a continuous homomorphism $h: \prod_{i=1}^n K_i \rightarrow T$ where each K_i is a compact chain and $Im(h) = T$.
- (8) Embedding dimension $(T, V) \leq n$ (in category \mathcal{L}).
- (9) If \hat{T} is the dual of (T, V) (i.e., the lattice of ideals of (T, V)), then \hat{T} as a \mathcal{J} -object has $SP(\hat{T}) \leq n$.

Proof (1) \Leftrightarrow (2) \Leftrightarrow (3) follows from definition of spread and Lemma 2.7 [H-SP].

(2) \Leftrightarrow (4) \Leftrightarrow (7) follows from Prop. 4.

(4) \Leftrightarrow (3) \Leftrightarrow (5) follows from Prop. 2.

(5) \Rightarrow (2) by Dilworth's theorem (width $\leq n \Rightarrow X = \bigcup_{i=1}^n C_i$) and Prop. 14 [H-SP].

(5) \Rightarrow (6) Immediate.

~~(1) \Rightarrow (7) By Lemma 2.7 [H-SP] \exists a continuous homomorphism $h: \prod_{i=1}^n K_i \rightarrow T$~~

~~where each K_i is a compact chain and $Im(h) = T$~~

(7) \Rightarrow (8) Take $\hat{h}: T \rightarrow \prod_{i=1}^n K_i$ where $\hat{h}(t) = \text{inf } h^{-1}(t)$

Then \hat{h} is one-to-one and preserves sups, hence is an embedding of (T, \vee) .

(8) \Rightarrow (9) By HMS duality $\exists h': \prod_{i=1}^n \hat{K}_i (= \prod_{i=1}^n \hat{K}_i)$

$\rightarrow \hat{T}$. K_i a chain $\Rightarrow \hat{K}_i$ is a compact chain.

Since (7) \Leftrightarrow (1), $Sp(\hat{T}) \leq n$.

(9) \Rightarrow (7) Again since (1) \Leftrightarrow (9), \exists compact chains

C_1, \dots, C_n and a continuous homomorphism $h: \prod_{i=1}^n C_i \rightarrow \hat{T}$

which is onto. By ATLAS $\exists p: \hat{T} \rightarrow T$, a continuous

homomorphism of \mathcal{CL} -objects. The composition $p \circ h$ gives (7).

COROLLARY 6 Let $T \in \mathcal{CL}$. Then

$$SP(T) = \text{Width}(\text{IRR } T) = \text{Embedding dim. } (T, \vee) = SP(\widehat{T, \vee})$$

if any of these quantities is finite.

COROLLARY 7 Let $T \in \mathcal{CL}$ and suppose T is distributive.

Then $SP(T) = Br(T)$.

Proof. $SP(T) \leq n \Rightarrow Br(T) \leq n$ by (7) of Thm. 5

since a homomorphism cannot raise breadth. Hence

$Br(T) \leq SP(T)$ always.

Conversely in the distributive case if

$Br(T) \leq n$, it is not difficult to show (see (8) of JDLI or

Lemma 3.2 of H-SP) that $\text{Width}(\text{PRIME } T) \leq n$. Since

$\text{PRIME } T = \text{IRR } T$, $SP(T) \leq n$ by (6) of Thm 5. Hence $SP(T) \leq Br(T)$.

~~Jim Lea~~ Jim Lea writes that he and K. Baker considered some related notions (mainly for distributive lattices). I summarize their definitions and Baker's results. (I do not have the proofs). Let (L, \wedge, \vee) be a lattice.

- (1) order dimension $(L) \equiv$ The least cardinal number of linear order relations on L whose intersection, as a subset of $L \times L$ is the given order relation on L
 $=$ least cardinal for which L as a poset can be embedded in product of chains of that cardinality
- (2) Embedding dimension $(L) \equiv$ The least cardinal of chains in which (L, \vee) can be embedded as a sublattice (or a sublattice if L is distributive).
- (3) Breadth (L)
- (4) $\text{Cov}(L)$, the covering number of $L \equiv$ Greatest number of elements which cover any single element (b covers a if $[a, b] = \{a, b\}$)
- (5) width $(\text{IRR } L)$
- (6) width $(\text{PI}(L)) \equiv$ width of prime ideal space
 $=$ width of $\text{PRIME}(L, \vee)$
- ~~(7) subcubic dimension $(L) \equiv$ sup of all $n \ni L$ contains a sublattice isomorphic to 2^n~~
- (7) orthodimension $(L) \equiv$ sup of all $n \ni \exists n$ distinct elements x_1, \dots, x_n and an element $c \ni x_i \wedge x_j = c$ for $i \neq j$.
- (9) Width $(\text{Join irreducibles})$.

Results:

- ① For ~~IRR~~ distributive lattices (1), (2), (3), (7), (8) are equivalent if all are finite
- ② All are equivalent for finite distributive lattices and apparently (except for 6) for dist. 3-objects (if all are finite).