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THE LARGE SCALE BEHAVIOR OF SUPER-BROWNIAN MOTION IN THREE DIMENSIONS WITH A SINGLE POINT SOURCE *

KLAUS FLEISCHMANN, CARL MUELLER, AND PASCAL VOGT

ABSTRACT. In a recent work, Fleischmann and Mueller (2004) showed the existence of a super-Brownian motion in $\mathbb{R}^d$, $d = 2, 3$, with extra birth at the origin. Their construction made use of an analytical approach based on the fundamental solution of the heat equation with a one point potential worked out by Albeverio et al. (1995). The present note addresses two properties of this measure-valued process in the three-dimensional case, namely the scaling of the process and the large scale behavior of its mean.

1. Introduction and result

A super-Brownian motion in $\mathbb{R}$ with a single point source δ_0 was constructed in Engländer & Fleischmann [3]. It was shown that its expected mass grows exponentially in time, and is in the mass-rescaled limit distributed in space as $x \mapsto e^{-|x|}$. In Engländer & Turaev [4] it is even proved that the random measures themselves grow in law exponentially as time increases, and are otherwise in the mass-rescaled limit spatially situated with the same shape except an overall random factor. The probabilistic effect behind the non-trivial existence of the model is the fact that a Brownian particle in $\mathbb{R}$ hits the origin with certainty and that it has there a non-degenerate local time, serving as an additional birth rate for the random creation of mass.

In higher dimensions, a Brownian particle fails to hit the origin, and a local time would degenerate. Nevertheless, Fleischmann & Mueller [6] succeeded in constructing a super-Brownian motion in $\mathbb{R}^d$, $d = 2, 3$, with a single point source. They heavily used well-known analytical facts from mathematical physics concerning Laplace operators with one-point-potentials. Heuristically, some additional rescaling enters the regularization of the delta function (serving as single point source). Properties of this new super-Brownian motion are not known so far. The *purpose* of the present note is to get some progress by studying its scaling and the large scale behavior of its expectation in the three-dimensional case.

1.1. The heat equation with one-point-potential. The Schrödinger equation with a one-point-potential is studied in quantum theory to describe singular

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electromagnetic effects on quantum particles, see e.g. the monograph Albeverio et al. [2, Part I]. By analytic continuation, solutions to the Schrödinger equation can be (at least formally) obtained via solutions of the heat equation.

Formally, the heat equation with a one-point-potential is given by

$$\partial_t u = \Delta u + \delta_0^{(\alpha)} u =: \Delta^{(\alpha)} u, \quad (1.1)$$

where ∂_t denotes the derivative with respect to time, Δ is the d -dimensional Laplacian, and $u : (0, \infty) \times \dot{\mathbb{R}}^d \rightarrow \mathbb{R}_+$ is a time-space field, where $\dot{\mathbb{R}}^d := \mathbb{R}^d \setminus \{0\}$ with the Euclidean metric is locally compact. If we denote by $B_\varepsilon(y)$ an open ball around $y \in \mathbb{R}^d$ of radius $\varepsilon > 0$, then having in mind that $\varepsilon^{-d} \mathbf{1}_{B_\varepsilon(0)} \approx \delta_0$, the operator $\Delta^{(\alpha)} := \Delta + \delta_0^{(\alpha)}$ is heuristically the limit as $\varepsilon \downarrow 0$ of the operator

$$\Delta_\varepsilon^{(\alpha)} := \Delta + h(d, \alpha, \varepsilon) \varepsilon^{-d} \mathbf{1}_{B_\varepsilon(0)}, \quad (1.2)$$

where $h(d, \alpha, \varepsilon)$ is some additional rescaling factor which depends on a parameter α at least. Restricting to $d = 3$, the function h can be chosen as

$$h(3, \alpha, \varepsilon) := \frac{\pi^2}{4} \varepsilon - 8\pi^2 \alpha \varepsilon^2, \quad \alpha \in \mathbb{R}, \quad \varepsilon > 0, \quad (1.3)$$

(cf. [2, (H.74)]).

Physically, α in the case $\alpha < 0$ is related to the *scattering length* $\text{sl}_\alpha := -(4\pi\alpha)^{-1}$ of the free Laplace operator Δ with respect to the interaction Laplacian $\Delta^{(\alpha)}$. Roughly speaking, the scattering length describes the average distance a free particle manages to go before any interaction takes place. So, if $\alpha \downarrow -\infty$ the scattering length $\text{sl}_\alpha \downarrow 0$ becomes smaller and we expect more interaction. For $\alpha \geq 0$ there is *no* proper physical interpretation of sl_α as the point spectrum of $\Delta^{(\alpha)}$ is empty (see [2, Theorem I.1.4]).

The *fundamental solution* p^α to the equation

$$\partial_t u = \Delta^{(\alpha)} u \quad \text{on } (0, \infty) \times \dot{\mathbb{R}}^d, \quad d = 2, 3, \quad (1.4)$$

which provides the basis for the analytical construction of the superprocess in [6], has been computed in Albeverio et al. [1]. In $d = 3$ (the two-dimensional case is more delicate, which is the reason we restrict to $d = 3$), the *one-point-interaction heat kernel* p^α for $\alpha \in \mathbb{R}$ is given by

$$\begin{aligned} p_t^\alpha(x, y) &= p_t(x, y) + \frac{2t}{|x||y|} p_t(|x| + |y|) - \frac{8\pi\alpha t}{|x||y|} \int_0^\infty du e^{-4\pi\alpha u} p_t(u + |x| + |y|), \end{aligned} \quad (1.5)$$

$t > 0$, $x, y \in \dot{\mathbb{R}}^3$, where p is the usual *free* heat kernel defined by,

$$p_t(x, y) := (4\pi t)^{-d/2} \exp(-|y - x|^2/4t), \quad (1.6)$$

and with a slight abuse of notation,

$$p_t(r) := (4\pi t)^{-d/2} \exp(-r^2/4t), \quad t > 0, \quad r \geq 0. \quad (1.7)$$

Also recall the scaling of the free heat kernel, i.e. for all $k, t > 0$ and $x, y \in \mathbb{R}^d$,

$$p_t(x, y) = k^{d/2} p_{kt}(k^{1/2}x, k^{1/2}y). \quad (1.8)$$

Note, that the last term in (1.5) is always finite and disappears for $\alpha = 0$. Moreover, $\alpha \mapsto p^\alpha$ is pointwise continuous and decreasing, and we have the (pointwise)

convergences $p^\alpha \uparrow +\infty$ as $\alpha \downarrow -\infty$ (i.e. the fundamental solution explodes which can be interpreted as *immediate interaction*), whereas $p^\alpha \downarrow p$ as $\alpha \uparrow +\infty$ leads the free case (i.e. the interaction disappears).

Rigorously, the family $\{\Delta^{(\alpha)} : \alpha \in \mathbb{R}\}$ of operators are defined as *all* self-adjoint extensions on the Hilbert space $\mathcal{L}^2(\dot{\mathbb{R}}^d, dx)$ of the Laplacian Δ acting on $\mathcal{C}_{\text{com}}^\infty(\dot{\mathbb{R}}^d)$, the space of infinitely differentiable functions on $\dot{\mathbb{R}}^d = \mathbb{R}^d \setminus \{0\}$ with compact support (see e.g. [2, Chapters I.1 and I.5]). Hence, although the p^α from (1.5) differ from the free heat kernel p , they solve the heat equation

$$\partial_t p_t^\alpha(x, y) = \Delta p_t^\alpha(x, y) \quad \text{on } (0, \infty) \times \dot{\mathbb{R}}^3, \quad (1.9)$$

with the Laplacian Δ acting either on the variable x or y . In particular, $(t, x, y) \mapsto p_t^\alpha(x, y)$ is jointly continuous on $(0, \infty) \times \dot{\mathbb{R}}^3 \times \dot{\mathbb{R}}^3$. Let us denote by S^α the semigroup associated with the kernel p^α , i.e.

$$S_t^\alpha \varphi(x) := \int_{\dot{\mathbb{R}}^3} dy \varphi(y) p_t^\alpha(x, y). \quad (1.10)$$

Note that S^α is *not* a contraction semigroup and so there is *no* stochastic process generated by this flow. The following lemma shows that in the present three-dimensional case the kernel p^α has a similar scaling behavior as the free heat kernel p .

Lemma 1.1 (Scaling of the p^α). *We have, for all $k, t > 0$, $x, y \in \dot{\mathbb{R}}^3$, and $\alpha \in \mathbb{R}$,*

$$p_t^\alpha(x, y) = k^{3/2} p_{kt}^{k^{-1/2}\alpha}(k^{1/2}x, k^{1/2}y). \quad (1.11)$$

Proof. It follows immediately from the definition (1.5) of the p^α and the scaling (1.8) of the free heat kernel p . $\square$

1.2. The flow associated with the one-point-interaction heat kernel.

This section is devoted to introducing a space of functions Φ on which the flow S^α acts as a strongly continuous linear semigroup (see [6, Section 2] for details). Let ϕ denote the *weight and reference function*

$$\phi(x) := |x|^{-1}, \quad x \in \dot{\mathbb{R}}^3 = \mathbb{R}^3 \setminus \{0\}. \quad (1.12)$$

For fixed $\varrho \in (1, 2)$, let $\mathcal{H} = \mathcal{H}^\varrho$ denote the space of measurable functions φ on $\dot{\mathbb{R}}^3$ for which

$$\|\varphi\|_{\mathcal{H}} := \left(\int_{\dot{\mathbb{R}}^3} dx \phi(x) |\varphi(x)|^\varrho \right)^{1/\varrho} < \infty. \quad (1.13)$$

Then $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is a Banach space, where as usual we do not distinguish between equivalence classes and their representatives. Now, let $\Phi = \Phi^\varrho$ denote the set of all *continuous* functions $\varphi : \dot{\mathbb{R}}^3 \rightarrow \mathbb{R}$ such that $\varphi \in \mathcal{H}$ and

$$0 \leq \varphi \leq C \phi \quad \text{for some constant } C = C_\varphi > 0. \quad (1.14)$$

We endow Φ with the topology inherited from $\mathcal{H}$. Note that the set $\mathcal{C}_{\text{com}}^+ = \mathcal{C}_{\text{com}}^+(\dot{\mathbb{R}}^3)$ of all non-negative, continuous functions on $\dot{\mathbb{R}}^3$ with compact support is contained in Φ . We remark that $\varphi \in \Phi$ might have a singularity at $x = 0$ of order $|x|^{-\xi}$ with $0 < \xi < 1$. The linear semigroup S^α introduced in (1.10) is strongly continuous on the cone $\Phi = \Phi^\varrho$, cf. Corollary 2.12 in [6].

1.3. Super-Brownian motion with a single point source. The superprocess X we want to study was constructed in [6] via the so-called *log-Laplace approach*. Roughly speaking, the Markov process X is uniquely determined by its log-Laplace transition functionals (s. (1.15) below), and they are described by a function v , the so-called log-Laplace function. The point is, that v solves uniquely a non-linear equation, the so-called log-Laplace equation (s. (1.16) below). The main work in [6] was, to verify that the Cauchy problem for this equation is well-posed. Here uniqueness followed from a contraction argument, existence was shown via a Picard iteration, and non-negativity of v followed using an approximating linearized equation. This then allows to construct X via a Trotter product approach to the related log-Laplace semigroup, more precisely, via an approximating log-Laplace equation related to separating critical continuous-state branching and mass flow according to S^α on alternate small time intervals.

Now we come to a precise description of X . Denote by $\mathcal{M} = \mathcal{M}(\dot{\mathbb{R}}^3)$ the set of all (Radon) measures μ on $\dot{\mathbb{R}}^3$ such that $\langle \mu, \varphi \rangle := \int_{\dot{\mathbb{R}}^3} \mu(dx) \varphi(x) < \infty$ for all $\varphi \in \Phi$. Recalling that $\mathcal{C}_{\text{com}}^+ \subset \Phi$, endow $\mathcal{M}$ with the vague topology.

Fix a constant $\eta > 0$ (branching rate). Suppose $0 < \beta < 1$ (branching index; the finite variance branching case $\beta = 1$ has been excluded in [6] for $d = 3$ for technical reasons). Then for each $\alpha \in \mathbb{R}$, there is a non-degenerate $\mathcal{M}$ -valued (time-homogeneous) Markov process $X = X^\alpha$ such that for (deterministic) starting measures $\mu \in \mathcal{M}$ and for $\varphi \in \Phi$,

$$-\log \mathbb{E}_\mu \exp \langle X_t^\alpha, -\varphi \rangle = \langle \mu, v(t, \cdot) \rangle, \quad t > 0, \quad (1.15)$$

where $v = \{v(t, x) : t \geq 0, x \in \dot{\mathbb{R}}^3\}$ is the unique non-negative solution of the integral equation related to the Φ -valued evolution equation

$$\begin{cases} \partial_t v = \Delta^{(\alpha)} v - \eta v^{1+\beta} & \text{on } (0, \infty), \\ v(0+, \cdot) = \varphi \end{cases} \quad (1.16)$$

(see [6, Theorem 4.4]). That is,

$$v(t, x) = \int_{\dot{\mathbb{R}}^3} dy \, p_t^\alpha(x, y) \varphi(y) - \eta \int_0^t ds \int_{\dot{\mathbb{R}}^3} dy \, p_{t-s}^\alpha(x, y) v^{1+\beta}(s, y), \quad (1.17)$$

$t > 0$, $x \in \dot{\mathbb{R}}^3$. Clearly, the first moments of X^α are determined by the S^α flow to be

$$\mathbb{E}_\mu \langle X_t^\alpha, \varphi \rangle = \langle \mu, S_t^\alpha \varphi \rangle, \quad (1.18)$$

for all starting measures $\mu \in \mathcal{M}$, $t \geq 0$, and $\varphi \in \Phi$.

1.4. Large scale behavior of the mean. Before we can state our result, we have to introduce some notation. The limiting measure will be expressed by means of the kernel

$$\vartheta_t^\alpha(x, y) := \frac{2t}{|x||y|} p_t(|y|) - \frac{8\pi\alpha t}{|x||y|} \int_0^\infty du \, e^{-4\pi\alpha u} p_t(u + |y|), \quad (1.19)$$

for $\alpha \in \mathbb{R}$, $t > 0$, and $x, y \in \dot{\mathbb{R}}^3$. Note that the integral is always finite, hence for $\alpha = 0$ the second term disappears. Moreover, the kernel ϑ^α is always non-negative.

This holds trivially whenever $\alpha < 0$, and to see this for $\alpha > 0$, use the estimate

$$p_t(u + |y|) \leq p_t(|y|). \quad (1.20)$$

We extend the definition of ϑ^α by setting

$$\vartheta_t^\alpha(x, y) := \begin{cases} 0, & \text{if } \alpha = +\infty, \\ +\infty, & \text{if } \alpha = -\infty. \end{cases} \quad (1.21)$$

The so defined kernels ϑ^α turn out to be pointwise continuous in $\alpha \in [-\infty, +\infty]$ (which follows from the arguments of the proof of Theorem 1.2 in Section 2.2 below).

Theorem 1.2 (Large scale behavior of the mean). *For $t > 0$, $\alpha, \lambda_k \in \mathbb{R}$, and all starting measures $X_0^\alpha = \mu \in \mathcal{M}$ satisfying $\langle \mu, \phi \rangle < \infty$, we have the convergence in $\mathcal{M}$,*

$$\lim_{k \uparrow \infty} k^{-1/2} \mathbb{E}_\mu [X_{kt}^{\lambda_k \alpha} (k^{1/2} dy)] = \langle \mu, \vartheta_t^{\alpha^*}(\cdot, y) \rangle dy, \quad (1.22)$$

provided that $\alpha^* := \lim_{k \uparrow \infty} k^{1/2} \lambda_k \alpha \in [-\infty, +\infty]$.

Note that in the special case $\lambda_k = k^{-1/2}$ we have $\alpha^* = \alpha \in \mathbb{R}$, whereas the particular case $\alpha = 0$ implies $\alpha^* = 0$ giving the simplified ϑ^0 in the limit term (without the second term in (1.19)).

Remark 1.3 (Large scale total mass). Taking $\lambda_k \equiv 1$ and insert formally $\varphi = 1$ as test function into convergence statement (1.22) yields

$$\lim_{k \uparrow \infty} k^{-1/2} \mathbb{E}_\mu \langle X_{kt}^\alpha, 1 \rangle = \begin{cases} 0 & \text{if } \alpha > 0, \\ 2t \langle \mu, \phi \rangle \int_{\mathbb{R}^3} dy \frac{1}{|y|} p_t(y) & \text{if } \alpha = 0, \\ \infty & \text{if } \alpha < 0. \end{cases} \quad (1.23)$$

A rigorous argument can be given along the same lines as the proof of Theorem 1.2 below. $\diamond$

1.5. Discussion and open problems. Let us comment on the *three cases* $\alpha^* = +\infty$, $\alpha^* \in \mathbb{R}$, and $\alpha^* = -\infty$ in Theorem 1.2. In the first case, the limiting mass disappears, more precisely, the scaled expression $\mathbb{E}_\mu [X_{kt}^{\lambda_k \alpha} (k^{1/2} dy)]$ is of order $o(k^{1/2})$. Roughly speaking, if $\alpha^* = +\infty$, then there are no interactions in the scaling limit (free case). In the second case, $\alpha^* \in \mathbb{R}$, the former expectation is about $k^{1/2} \langle \mu, \vartheta_t^{\alpha^*}(\cdot, y) \rangle dy$. Note that these measures are decreasing in α^* . Finally, if $\alpha^* = -\infty$, we have immediate interaction in the large scale limit leading to the explosion of the expected mass.

Clearly, to describe only the large scale behavior of the *expected processes* is unsatisfactory. It is desirable to get insight into the processes themselves. Recall that in the one-dimensional case the large time behavior in law of the process itself is known from [4] (for a sharpening of some results from [4], see Engländer & Winter [5]). However, we stress the fact, that the process in three dimensions is expected to have quite different features. For instance, if $\alpha = 0$, then according to Remark 1.3 the total mass grows with a power order, whereas in one dimension the growth is exponential. Moreover, in the three-dimensional case one needs additionally to contract the normalized measures to get a limit. For the measures

themselves, scaled as in Theorem 1.2, *there might be extinction in law* despite convergence of their expectations as in (1.22).

Another *open problem* is the large scale behavior of $\mathbb{E}X^\alpha$ in the two-dimensional case, in which the fundamental solutions p^α from [1] are analytically more delicate, see e.g. [6, formula (2.30)]. In particular, a scaling property as in Lemma 1.1 is not available.

Remark 1.4 (Discrete version). In physics literature, Redner & Kang [7], the following somehow related model occurs: Discrete time random walkers produce a fixed number of additional particles if they hit the origin in $\mathbb{Z}^d$. Dimension effects occur. $\diamond$

2. Proofs

2.1. A scaling property. Recall that we are dealing with the three-dimensional case.

Proposition 2.1 (A scaling property). *Let $t, k > 0$, $\mu \in \mathcal{M}$, and $\alpha, \lambda_k \in \mathbb{R}$. Then*

$$\begin{aligned} & \left\{ k^{-1/\beta} X_{kt}^{\lambda_k \alpha}(k^{1/2} \cdot) \mid X_0^{\lambda_k \alpha} = k^{1/\beta} \mu(k^{-1/2} \cdot) \right\} \\ & \stackrel{\mathcal{L}}{=} \left\{ X_t^{k^{1/2} \lambda_k \alpha} \mid X_0^{k^{1/2} \lambda_k \alpha} = \mu \right\}. \end{aligned} \quad (2.1)$$

Note that besides $\lambda_k \equiv 1$, the cases $\lambda_k = k^{-1/2}$ or even $\alpha = 0$ are particularly nice, since here the right hand side in (2.1) is independent of k (a kind of self-similarity).

Proof. For $\varphi \in \Phi$ fixed,

$$\langle k^{-1/\beta} X_{kt}^{\lambda_k \alpha}(k^{1/2} dy), \varphi \rangle = \langle X_{kt}^{\lambda_k \alpha}, k^{-1/\beta} \varphi(k^{-1/2} \cdot) \rangle, \quad (2.2)$$

hence, by (1.15) and (1.17),

$$\begin{aligned} & -\log \mathbb{E}_{k^{1/\beta} \mu(k^{-1/2} \cdot)} \exp \langle k^{-1/\beta} X_{kt}^{\lambda_k \alpha}(k^{1/2} dy), -\varphi \rangle \\ & = -\log \mathbb{E}_{k^{1/\beta} \mu(k^{-1/2} \cdot)} \exp \langle X_{kt}^{\lambda_k \alpha}, -k^{-1/\beta} \varphi(k^{-1/2} \cdot) \rangle \\ & = \langle k^{1/\beta} \mu(k^{-1/2} \cdot), v(kt, \cdot) \rangle = \langle \mu, k^{1/\beta} v(kt, k^{1/2} \cdot) \rangle, \end{aligned} \quad (2.3)$$

where $\{v(t', x') : t' \geq 0, x' \in \dot{\mathbb{R}}^3\}$ is the non-negative solution of the integral equation related to the function-valued evolution equation

$$\begin{cases} \partial_t v = \Delta^{(\lambda_k \alpha)} v - \eta v^{1+\beta} & \text{on } (0, \infty), \\ v(0+, \cdot) = k^{-1/\beta} \varphi(k^{-1/2} \cdot). \end{cases} \quad (2.4)$$

More precisely,

$$\begin{aligned} k^{1/\beta} v(kt, k^{1/2} x) &= k^{1/\beta} \int_{\dot{\mathbb{R}}^3} dy p_{kt}^{\lambda_k \alpha}(k^{1/2} x, y) k^{-1/\beta} \varphi(k^{-1/2} y) \\ &\quad - k^{1/\beta} \eta \int_0^{kt} ds \int_{\dot{\mathbb{R}}^3} dy p_{kt-s}^{\lambda_k \alpha}(k^{1/2} x, y) v^{1+\beta}(s, y). \end{aligned} \quad (2.5)$$

By a change of variable,

$$\begin{aligned} k^{1/\beta} v(kt, k^{1/2}x) &= \int_{\mathbb{R}^3} dy \, k^{3/2} p_{kt}^{\lambda_k \alpha}(k^{1/2}x, k^{1/2}y) \varphi(y) \\ &\quad - k^{1/\beta} \eta \int_0^t ds \, k \int_{\mathbb{R}^3} dy \, k^{3/2} p_{kt-ks}^{\lambda_k \alpha}(k^{1/2}x, k^{1/2}y) v^{1+\beta}(ks, k^{1/2}y). \end{aligned} \quad (2.6)$$

Hence, by Lemma 1.1,

$$\begin{aligned} k^{1/\beta} v(kt, k^{1/2}x) &= \int_{\mathbb{R}^3} dy \, p_t^{k^{1/2} \lambda_k \alpha}(x, y) \varphi(y) - k^{1/\beta} \eta \int_0^t ds \, k \int_{\mathbb{R}^3} dy \, p_{t-s}^{k^{1/2} \lambda_k \alpha}(x, y) v^{1+\beta}(ks, k^{1/2}y). \end{aligned} \quad (2.7)$$

Since $1/\beta + 1 - (1/\beta)(1+\beta) = 0$ we see that $k^{1/\beta} v(kt, k^{1/2}x) =: w_k(t, x)$ satisfies the equation

$$w_k(t', x') = \int_{\mathbb{R}^3} dy \, p_{t'}^{k^{1/2} \lambda_k \alpha}(x', y) \varphi(y) - \eta \int_0^{t'} ds \int_{\mathbb{R}^3} dy \, p_{t'-s}^{k^{1/2} \lambda_k \alpha}(x', y) w_k^{1+\beta}(s, y),$$

$t' > 0$, $x' \in \mathbb{R}^3$. By uniqueness of solutions of the log-Laplace equation (1.17) and by (1.15), claim (2.1) follows. $\square$

2.2. Proof of Theorem 1.2. Fix $\varphi \in \mathcal{C}_{\text{com}}^+(\mathbb{R}^3)$. Using formula (1.18) for the first moment of X^α and substitution, we obtain

$$\begin{aligned} k^{-1/2} \mathbb{E}_\mu \langle X_{kt}^{\lambda_k \alpha}, \varphi(k^{-1/2} \cdot) \rangle &= k^{-1/2} \int_{\mathbb{R}^3} \mu(dx) \int_{\mathbb{R}^3} dy \, p_{kt}^{\lambda_k \alpha}(x, y) \varphi(k^{-1/2}y) \\ &= k^{-1/2} \int_{\mathbb{R}^3} \mu(dx) \int_{\mathbb{R}^3} dy \, k^{3/2} p_{kt}^{\lambda_k \alpha}(x, k^{1/2}y) \varphi(y). \end{aligned} \quad (2.8)$$

By Lemma 1.1 this is equal to

$$k^{-1/2} \int_{\mathbb{R}^3} \mu(dx) \int_{\mathbb{R}^3} dy \, p_t^{k^{1/2} \lambda_k \alpha}(k^{-1/2}x, y) \varphi(y). \quad (2.9)$$

Using formula (1.5) for the expression of p^α , we get three terms, we will deal with separately.

1° (*First term*). The first term equals,

$$k^{-1/2} \int_{\mathbb{R}^3} \mu(dx) \int_{\mathbb{R}^3} dy \, p_t(k^{-1/2}x, y) \varphi(y). \quad (2.10)$$

This double integral is finite and vanishes as $k \uparrow \infty$. To see this, let us restrict the outer integral first to $|x| > K$ where we specify $K \geq 1$ later. We call this restricted integral I_K . We use $\varphi \leq C\phi$ (since $\varphi \in \Phi$) and, with S denoting the free heat flow,

$$S_t \phi \leq C\phi, \quad t \geq 0, \quad (2.11)$$

with changed constant C (see [6, Lemma 2.1]) to arrive at

$$I_K \leq C k^{-1/2} \int_{|x|>K} \mu(dx) \phi(k^{-1/2}x) = C \int_{|x|>K} \mu(dx) \phi(x), \quad (2.12)$$

the last step by the particular form of ϕ . The latter integral can be made arbitrarily small (uniformly in k) by choosing K sufficiently large (by our assumption on μ).

It remains to deal with the case $|x| \leq K$ for fixed K . We split the internal integral in (2.10) as follows. First, if $|k^{-1/2}x - y| \geq |y|/2$, then

$$p_t(k^{-1/2}x, y) \leq p_t(|y|/2), \quad (2.13)$$

which leads to the bound

$$k^{-1/2} \int_{|x| \leq K} \mu(dx) \int_{\mathbb{R}^3} dy p_t(|y|/2) \varphi(y) \longrightarrow 0 \quad \text{as } k \uparrow \infty \quad (2.14)$$

(the $\mu(dx)$ -integral is finite as $\langle \mu, \phi \rangle < \infty$). On the other hand, if $|k^{-1/2}x - y| < |y|/2$, then $-k^{-1/2}|x| + |y| < |y|/2$ which implies $|y| < 2k^{-1/2}|x| \leq 2K$. Hence as $p_t(k^{-1/2}x, y) \leq Ct^{-3/2}$ and $\varphi \leq C\phi$, we get the upper estimate

$$C_t k^{-1/2} \int_{|x| \leq K} \mu(dx) \int_{|y| < 2K} dy \phi(y) \longrightarrow 0 \quad \text{as } k \uparrow \infty \quad (2.15)$$

(the dy -integral is finite, since we are in dimension three).

2° (*Second term*). The second term reads

$$\int_{\mathbb{R}^3} \mu(dx) \int_{\mathbb{R}^3} dy \frac{2t}{|x||y|} p_t(k^{-1/2}|x| + |y|) \varphi(y) =: II_k. \quad (2.16)$$

Observe that,

$$p_t(k^{-1/2}|x| + |y|) \uparrow p_t(|y|) \quad \text{as } k \uparrow \infty. \quad (2.17)$$

We can apply the monotone convergence theorem to obtain the limit,

$$\lim_{k \uparrow \infty} II_k = \int_{\mathbb{R}^3} \mu(dx) \frac{2t}{|x|} \int_{\mathbb{R}^3} dy \frac{1}{|y|} p_t(y) \varphi(y), \quad (2.18)$$

where finiteness follows from $\langle \mu, \phi \rangle < \infty$ and $\varphi \leq C\phi$. Hence, this summand gives the first part of the kernel ϑ^{α^*} .

3° (*Third term*). It remains to insert the scaled third term from representation (1.5) into (2.9) which reads as

$$k^{-1/2} \int_{\mathbb{R}^3} \mu(dx) \int_{\mathbb{R}^3} dy \frac{-8\pi t k^{1/2} \lambda_k \alpha}{k^{-1/2}|x||y|} \int_0^\infty du e^{-4\pi k^{1/2} \lambda_k \alpha u} p_t(u + k^{-1/2}|x| + |y|) \varphi(y). \quad (2.19)$$

We distinguish several cases: If $\lambda_k \alpha = 0$ for all sufficiently large k , then the third term disappears and we are done. From now on assume $\lambda_k \alpha \neq 0$ for all k . Substituting $u \mapsto 4\pi k^{1/2} |\lambda_k \alpha| u$ into (2.19) yields,

$$\int_{\mathbb{R}^3} \mu(dx) \int_{\mathbb{R}^3} dy \frac{-2t \operatorname{sign}(\lambda_k \alpha)}{|x||y|} \int_0^\infty du e^{-\operatorname{sign}(\lambda_k \alpha) u} p_t\left(\frac{u}{4\pi k^{1/2} |\lambda_k \alpha|} + k^{-1/2}|x| + |y|\right) \varphi(y). \quad (2.20)$$

Now let $k^{1/2} |\lambda_k \alpha| \rightarrow \infty$. We may consider a monotone subsequence of $k^{1/2} \lambda_k \alpha$. Clearly,

$$p_t\left(\frac{u}{4\pi k^{1/2} |\lambda_k \alpha|} + k^{-1/2}|x| + |y|\right) \uparrow p_t(|y|) \quad \text{as } k \uparrow \infty, \quad (2.21)$$

and by monotone convergence, (2.20) converges along the subsequence to

$$-\int_{\mathbb{R}^3} \mu(dx) \frac{2t \operatorname{sign}(\alpha^*)}{|x|} \int_0^\infty du e^{-\operatorname{sign}(\alpha^*)u} \int_{\mathbb{R}^3} dy \frac{1}{|y|} p_t(y) \varphi(y), \quad (2.22)$$

which is independent of the choice of the subsequence. Note that

$$\operatorname{sign}(\alpha^*) \int_0^\infty du e^{-\operatorname{sign}(\alpha^*)u} = \begin{cases} 1 & \text{if } \operatorname{sign}(\alpha^*) = 1, \\ +\infty & \text{if } \operatorname{sign}(\alpha^*) = -1. \end{cases} \quad (2.23)$$

In the first case the second and the third limiting terms cancel.

Next, let $k^{1/2}\lambda_k\alpha \rightarrow 0$. Note, that

$$p_t\left(\frac{u}{4\pi k^{1/2}|\lambda_k\alpha|} + k^{-1/2}|x| + |y|\right) \leq p_t\left(\frac{u}{4\pi k^{1/2}|\lambda_k\alpha|}\right) e^{-|y|^2/4t}. \quad (2.24)$$

In this case the double integral (2.20) is in absolute value bounded by

$$\int_{\mathbb{R}^3} \mu(dx) \int_{\mathbb{R}^3} dy \frac{2t}{|x||y|} e^{-|y|^2/4t} \varphi(y) \int_0^\infty du e^u p_t\left(\frac{u}{4\pi k^{1/2}|\lambda_k\alpha|}\right), \quad (2.25)$$

which tends to 0 as $k \uparrow \infty$ as the $\mu(dx)$ and dy -integrals are finite and in the du -integral the p_t -term compensates the e^u .

It remains to deal with the case $k^{1/2}\lambda_k\alpha \rightarrow \alpha^* \in \dot{\mathbb{R}}^1$. Note, that we only have to justify to change the limit and integration in (2.20), as substituting $u \mapsto (4\pi|\alpha^*|)^{-1}u$ leads to the desired expression. To justify the interchange, we estimate as in (2.24). The resulting $\mu(dx)$ and dy -integrals are independent of k and finite, whereas to dominate in the second integral we use $k^{1/2}|\lambda_k\alpha| \leq |\alpha^*| + 1$ for all sufficiently large k . $\square$

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