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## On a Class of Average Preserving Semi-Martingale Laws Optimization Problems

Rémi Lassalle

*Université Paris-Dauphine, PSL, Place du Maréchal De Lattre De Tassigny, 75775 Paris Cedex 16, France,*  
lassalle@ceremade.dauphine.fr

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## ON A CLASS OF AVERAGE PRESERVING SEMI-MARTINGALE LAWS OPTIMIZATION PROBLEMS

RÉMI LASSALLE\*

**ABSTRACT.** In this paper, within the specific framework of a recently developed information flows preserving calculus of variations, we investigate a class of average fixed optimization problems, over sets of laws of Itô semi-martingales. We show that, within those conditions, the optimums are clearly ruled by a least action principle, which yields the corresponding Euler-Lagrange conditions; this enlightens their specific dynamics. In particular, this encompasses a specific class of semi-martingale optimal transportation problems, and specific entropy minimization problems in close connection to the Schrödinger problem.

### 1. Introduction

In the 1930's, E. Schrödinger introduced (see [44],[45]) what is nowadays known as the *Schrödinger problem*. Although it was shed into light by who is usually known as a celebrated physicist, it was soon considered as a mathematical problem (see [6]). We refer to [21] and to [35] for synthetic introductions on this topic. In [21], a representation of the *relative entropy* with respect to the *Wiener measure* shows that the specific entropy minimization involved, can be seen equivalently as an action functional minimization. On the other hand, works in the line of [53], [54] have shown that these problems could be used to produce some mathematical tools which suggest a convincing analogy to classical mechanics; due to the specificities of the *Schrödinger problem*, this should be distinguished from works in the line of [8]. Ever since, there have been several contributions in this direction, among many see [37], [41], [42], [46], [48]. In [33], a specific calculus of variations on a precise set of laws of stochastic processes, was developed. It enables to recover, within this specific context, both the mathematical theorem of *calculus of variations* related to classical Hamilton's *least action principle* of physics (see [1], [3], [13], [24], [31]), and the critical conditions, under the form of [32], of *Schrödinger bridges*, as an application of a same theorem; the corresponding *Euler-Lagrange conditions* are established on the law of the whole process, so that they encapsulate the relevant information related to what can be interpreted as the dynamic of the system, within a model. This approach has several advantages. First, it is intrinsic, on the canonical space endowed with the law of the process, and does not depend on

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\* Corresponding author.

the choice of a stochastic basis where a specific model would be considered: this provides critical conditions which, for specific cost maps, yield specific *forward-backward systems* of *stochastic differential equations* (see [38]). Then, from its definition, it naturally provides rigorous and compact statements and proofs. This involves features of *filtering* ([4]) related to the *innovation conjecture* (see [5], [27]), in close connection with the celebrated Tsirelson counter-example (see [49]). Finally, from the definitions, this construction is essentially based on *information flows* preserving transports of measure, which are interesting by themselves in view of applications; here we have followed the terminology p. 39 of [14].

To be accurate, subsequently  $\mathbb{S}$  denotes a specific set of laws of  $\mathbb{R}^d$ -valued continuous *Itô semi-martingales*, while for any  $\nu \in \mathbb{S}$ ,  $L_a^2(\nu, H_{0,0}^1)$  denotes a specific set of absolutely continuous  $\mathbb{R}^d$ -valued adapted processes, with vanishing end-points, whose a.e. derivatives are square integrable; alternatively, the latter may also be seen as random functions. Both are recalled accurately below, together with the  $L_a^2(\nu, H_{0,0}^1)$ -differential. The drift and dispersion *local characteristics* of any  $\nu \in \mathbb{S}$  are denoted by  $(v_t^\nu, \alpha_t^\nu)$ ;  $(W_t)_{t \in [0,1]}$  denotes the evaluation process, and we note  $\mathbf{0}_{H^1} : t \in [0,1] \rightarrow \mathbf{0}_{\mathbb{R}^d} \in \mathbb{R}^d$ . Moreover, given a measurable cost map  $\mathcal{L}$ , the corresponding action  $\mathcal{S} : \mathbb{S} \rightarrow \mathbb{R} \cup \{+\infty\}$ , is defined by

$$\mathcal{S}(\nu) := \begin{cases} \mathbb{E}_\nu \left[ \int_0^1 \mathcal{L}_t(W_t, v_t^\nu, \alpha_t^\nu) dt \right], & \text{if } \mathbb{E}_\nu \left[ \int_0^1 |\mathcal{L}_t(W_t, v_t^\nu, \alpha_t^\nu)| dt \right] < +\infty \\ +\infty, & \text{otherwise} \end{cases}, \quad (1.1)$$

for all  $\nu \in \mathbb{S}$ . In the whole paper,  $\lambda$  denotes the *Lebesgue measure* on  $[0,1]$ .

In [34], within the specific framework of [33], the following average preserving least action principle has been investigated. Under conditions on  $\mathcal{L}$  and  $\nu \in \mathbb{S}$ , such that  $\mathcal{S}$  is  $L_a^2(\nu, H_{0,0}^1)$ -differentiable at  $\nu \in \mathbb{S}$ , we have

$$\delta \mathcal{S}_\nu[h] = 0, \quad \forall h \in L_a^2(\nu, H^1) : h_0 = h_1 = \mathbf{0}_{\mathbb{R}^d}, \quad \nu - a.s., \quad \text{and} \quad \int_W h \, d\nu = \mathbf{0}_{H^1},$$

where the latter is a *Bochner integral* (see [9]), if and only if, there exists a *càd-làg*  $(\mathcal{F}_t^\nu)$ -martingale  $(N_t^\nu)$  on  $(W, \mathcal{B}_W^\nu, \nu)$ , and a square integrable deterministic measurable process  $(A_t^\nu)$ , such that

$$\partial_\nu \mathcal{L}_t(W_t, v_t^\nu, \alpha_t^\nu) - \int_0^t \partial_x \mathcal{L}_s(W_s, v_s^\nu, \alpha_s^\nu) ds = A_t^\nu + N_t^\nu, \quad \lambda \otimes \nu - a.e., \quad (1.2)$$

where  $(\mathcal{F}_t^\nu)$  denotes the  $\nu$ -*usual augmentation* of the filtration generated by the evaluation process.

Then, [34] has provided a method to construct such critical processes with specific forward-backward systems. This paper provides results to obtain critical laws of this average preserving least action principle. As a byproduct, it also yields an approach to establish the existence of solutions to specific forward-backward systems investigated in [34].

The structure of this paper is as follows. In Section 2, we fix the notation and the framework. In particular we provide a recall on the definition of the *intrinsic stochastic derivative* of [33], and on the average preserving variation processes of [34]. We relate the critical points of this least action principle with average

preserving semi-martingale optimization problems in Section 3; it is related to a specific semi-martingale optimal transportation problem (see [46]). Let

$$\mathbb{S}_{\nu_0, \nu_1}^1 := \{\nu \in \mathbb{S} : \mathbb{E}_\nu [\|W_t\|_{\mathbb{R}^d}] < +\infty, \forall t \in [0, 1], W_{0*}\nu = \nu_0, W_{1*}\nu = \nu_1\},$$

where  $\nu_0, \nu_1 \in \mathbf{M}_1(\mathbb{R}^d)$  denote two Borel probability measures on  $\mathbb{R}^d$ , such that

$$\int_{\mathbb{R}^d} \|x\|_{\mathbb{R}^d} \nu_0(dx) + \int_{\mathbb{R}^d} \|x\|_{\mathbb{R}^d} \nu_1(dx) < +\infty,$$

and where  $W_{0*}\nu$  (resp.  $W_{1*}\nu$ ) denotes the initial (resp. final) marginal law of  $\nu \in \mathbb{S}$ . We provide informations on what we interpret as the dynamic of optimums to

$$\inf (\{\mathcal{S}(\nu) : \nu \in \mathbb{A}_{\nu_0, \nu_1}^{l, \alpha}\}),$$

where  $\mathbb{A}_{\nu_0, \nu_1}^{l, \alpha}$  denotes the subset

$$\left\{ \nu \in \mathbb{S}_{\nu_0, \nu_1}^1 : \mathbb{E}_\nu[W_t] = \mathbb{E}_\nu[W_0] + l(t), \forall t \in [0, 1], \int_0^\cdot \alpha_t^\nu dt = \int_0^\cdot \alpha_t dt, \nu - a.s. \right\},$$

by showing that under conditions, the optimums to those problems satisfy (1.2);  $(\alpha_t)_{t \in [0, 1]}$  is assumed to be obtained from a Borel measurable function  $\alpha : [0, 1] \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$  such that  $\alpha_t$  is symmetric non-negative for all  $t \in [0, 1]$ , and

$$\int_0^1 \|\alpha_t\|_{\mathbb{R}^d \otimes \mathbb{R}^d} dt < +\infty,$$

while  $l : [0, 1] \rightarrow \mathbb{R}^d$  is an element of the so-called *Cameron-Martin space* (see below). Furthermore we show that under conditions, it also holds with semi-martingale optimization problems of the specific form

$$\inf (\{\mathcal{S}(\nu) : \nu \in \mathbb{A}_\gamma^{l, \alpha}\}),$$

where  $\mathbb{A}_\gamma^{l, \alpha}$  denotes the subset

$$\left\{ \nu \in \mathbb{S}_\gamma^1 : \mathbb{E}_\nu[W_t] = \mathbb{E}_\nu[W_0] + l(t), \forall t \in [0, 1], \int_0^\cdot \alpha_t^\nu dt = \int_0^\cdot \alpha_t dt, \nu - a.s. \right\},$$

and where

$$\mathbb{S}_\gamma^1 := \{\nu \in \mathbb{S} : \mathbb{E}_\nu [\|W_t\|_{\mathbb{R}^d}] < +\infty, \forall t \in [0, 1], (W_0, W_1)_*\nu = \gamma\},$$

$\gamma \in \mathbf{M}_1(\mathbb{R}^d \times \mathbb{R}^d)$  denoting a Borel probability measure on  $\mathbb{R}^d \times \mathbb{R}^d$ . Finally, given specific cost functions, we investigate entropy minimization problems, closely related to the Schrödinger problem in Section 4. On the *Wiener space* context, the latter provide sufficient conditions of existence to laws which satisfy (1.2) with specific cost maps.

## 2. Preliminaries and Notation

**General notation.** In the whole paper, for any *Polish space*  $E$ , we denote by  $\mathbf{M}_1(E)$  the set of Borel probability measures on the measurable space  $(E, \mathcal{B}_E)$ , and given  $\nu \in \mathbf{M}_1(E)$ ,  $\mathcal{B}_E^\nu$  denotes the  $\nu$ -completion of the Borel sigma-field  $\mathcal{B}_E$ ; see [20] or [43]. One of those spaces which will be of specific interest is the space  $W := C([0, 1]; \mathbb{R}^d)$  of the continuous  $\mathbb{R}^d$ -valued functions on  $[0, 1]$ . It is endowed with the norm  $\|\cdot\|_W$  of uniform convergence.

As far as stochastic processes are involved, we work on complete stochastic basis  $(\Omega, \mathcal{A}, (\mathcal{A}_t)_{t \in [0,1]}, \mathcal{P})$ , i.e. the sigma-field  $\mathcal{A}$  is  $\mathcal{P}$ -complete, and the filtration  $(\mathcal{A}_t)$  satisfies the usual conditions (it is right-continuous and  $\mathcal{P}$ -complete, see [15]). Moreover, we use the usual notation  $\mathbb{E}_{\mathcal{P}}[X] := \sum_{i=1}^d \mathbb{E}_{\mathcal{P}}[X_i] e_i$ ,  $(e_i)_{i \in \{1, \dots, d\}}$  (resp.  $X_i$ ) denoting canonical basis of  $\mathbb{R}^d$  (resp. the  $i$ -th component of  $X$  in this basis), for any  $X \in L^1(\mathcal{P}, \mathbb{R}^d)$ . The evaluation  $(W_t)_{t \in [0,1]}$  is the process which to a given  $t \in [0,1]$ , and a given  $\omega \in W$ , associates the value at  $t$  of the function  $\omega$ , i.e.  $W_t(\omega) := \omega(t)$ . Given  $\nu \in \mathbf{M}_1(W)$ , we denote by  $(\mathcal{F}_t^\nu)$  the *usual augmentation* of the filtration generated by the evaluation process  $(W_t)_{t \in [0,1]}$  on the probability space  $(W, \mathcal{B}_W^\nu, \nu)$ . For any  $\nu \in \mathbf{M}_1(W)$ , we denote by  $M_\nu((W, \mathcal{B}_W^\nu), (W, \mathcal{B}_W))$  the set which is obtained by identifying the  $\mathcal{B}_W^\nu/\mathcal{B}_W$ -measurable functions  $f : W \rightarrow W$  which coincide  $\nu$ -almost everywhere. If  $U \in M_\nu((W, \mathcal{B}_W^\nu), (W, \mathcal{B}_W))$ ,  $(\mathcal{G}_t^U)$  denotes the  $\nu$ -usual augmentation of the filtration  $(\sigma(f_s, s \leq t))$ , for any  $\mathcal{B}_W^\nu/\mathcal{B}_W$ -measurable function  $f : W \rightarrow W$  whose  $\nu$ -equivalence class is  $U$ ; we have set  $f_s := W_s \circ f$ , for all  $s \in [0,1]$ . The *classical Cameron-Martin space*  $H^1$  is the *Hilbert space* defined by

$$H^1 := \left\{ h \in C([0,1]; \mathbb{R}^d) : h = \int_0^\cdot \dot{h}_s ds, \|h\|_{H^1}^2 := \int_0^1 \|\dot{h}_s\|_{\mathbb{R}^d}^2 ds < +\infty \right\},$$

and in view of variations preserving the initial and final marginals we will also use the space

$$H_{0,0}^1 := \{h \in H^1 : h_0 = \mathbf{0}_{\mathbb{R}^d}\};$$

recall that since  $h_0 = \mathbf{0}_{\mathbb{R}^d}$  for any  $h \in H^1$ , we have  $h_0 = h_1 = \mathbf{0}_{\mathbb{R}^d}$ , for all  $h \in H_{0,0}^1$ . The Cameron-Martin space is fundamental in stochastic analysis (see [39]) and takes its name from R.H. Cameron and W.T. Martin (for instance see [11], [12]), who, together with their collaborators, are known to have initiated advanced integral calculus in infinite dimension (see [23]).

The set  $L_a^2(\nu, H^1)$  (resp.  $L_a^2(\nu, H_{0,0}^1)$ ) is identified to the set of  $u \in M_\nu((W, \mathcal{B}_W^\nu), (W, \mathcal{B}_W))$ , which are  $(\mathcal{F}_t^\nu)$ -adapted (since  $(\mathcal{F}_t^\nu)$  satisfies the usual conditions, it is equivalent to  $(\mathcal{G}_t^u) \subset (\mathcal{F}_t^\nu)$ ) and which further satisfy  $\mathbb{E}_\nu[\|u\|_{H^1}^2] < +\infty$  (resp.  $\mathbb{E}_\nu[\|u\|_{H^1}^2] < +\infty$  and  $u_0 = u_1 = \mathbf{0}_{\mathbb{R}^d}$ ,  $\nu - a.s.$ ).

**Information flow preserving maps.** Given  $\nu \in \mathbf{M}_1(W)$ , in this paper, we call an *information flow preserving map*, or an isomorphism of filtered probability space, any  $U \in M_\nu((W, \mathcal{B}_W^\nu), (W, \mathcal{B}_W))$  which satisfies

$$(\mathcal{G}_t^U) = (\mathcal{F}_t^\nu).$$

Here, we follow the same terminology as [14] p. 39, which interprets a filtration as an information flow. We refer to [33] for much details on those mathematical objects, which appear as fundamental tools in stochastic analysis (see [39]); we denote by  $\mathcal{I}_f(\nu)$  the set of those isomorphisms of filtered probability spaces of  $(W, \mathcal{B}_W^\nu, (\mathcal{F}_t^\nu), \nu)$  to any Borel probability measure on  $W$ . For the sake of efficiency, since  $\mathcal{F}_0^\nu$  does not necessarily coincide with  $\sigma(W_0)^\nu$ , we also define the set  $\mathcal{I}_f^0(\nu)$  by

$$\mathcal{I}_f^0(\nu) := \{U \in \mathcal{I}_f(\nu) : \sigma(W_0)^\nu = \sigma(U_0)^\nu\},$$

which we interpret as the set of information flow preserving maps, which preserve the initial information.

**Average preserving variation processes.** Recall that given  $\nu \in \mathbf{M}_1(W)$ , the set of *variation processes*  $V_\nu$  at  $\nu$  was defined in [33] to be the subset

$$\{h \in L_a^2(\nu, H^1) : U \in \mathcal{I}_f^0(\nu) \implies U + h \in \mathcal{I}_f^0(\nu), \forall U \in M_\nu((W, \mathcal{B}_W^\nu), (W, \mathcal{B}_W))\};$$

recall that  $V_\nu$  is a vector space (see [33]). Those may be interpreted as perturbations, ruled by an absence of information loss principle, which preserve information flows. Moreover, we will use the set  $V_\nu^\infty$  of the  $h \in V_\nu$  such that there exists a  $C > 0$  which meets  $\|h\|_W \leq C$ ,  $\nu$ -a.s.. In view of applying least action principles, the subset  $V_\nu^{0,\infty}$  of the  $h \in V_\nu^\infty$  such that

$$h_0 = h_1 = \mathbf{0}_{\mathbb{R}^d}, \quad \nu - a.s.,$$

and the subset  $A_\nu^{0,\infty}$  of *average preserving variation processes* at  $\nu \in \mathbf{M}_1(W)$ , which was defined in [34] by

$$A_\nu^{0,\infty} := \left\{ h \in V_\nu^{0,\infty} : \int_W h \, d\nu = \mathbf{0}_{H^1} \right\}, \quad (2.1)$$

will be used subsequently. It can be seen (see [34]) that it is a dense subspace of

$$\left\{ h \in L_a^2(\nu, H_{0,0}^1) : \int_W h \, d\nu = \mathbf{0}_{H^1} \right\},$$

relatively to the  $L_a^2(\nu, H_{0,0}^1)$  topology.

**The differential of [33].** By following [33], we say that a function

$$\phi : \nu \in \mathbf{M}_1(W) \rightarrow \phi(\nu) \in \mathbb{R} \cup \{+\infty\},$$

is  $L_a^2(\nu, H_{0,0}^1)$ -differentiable at  $\nu \in \mathbf{M}_1(W)$ , if  $\phi(\nu) < +\infty$ , if  $\frac{d}{d\epsilon} \phi(\nu^{\epsilon k})|_{\epsilon=0}$  exists for all  $k \in V_\nu^{0,\infty}$ , and there exists a  $\xi \in L_a^2(\nu, H_{0,0}^1)$  such that

$$\frac{d}{d\epsilon} \phi(\nu^{\epsilon k})|_{\epsilon=0} = \int_W \langle \xi, k \rangle_{H^1} \, d\nu,$$

for all  $k \in V_\nu^{0,\infty}$ , where  $\nu^{\epsilon k} := (I_W + \epsilon k)_\star \nu$ ;  $I_W$  denotes the identity map on  $W$ , and  $\star$ , the push-forward of measure. In this case, we define

$$\delta\phi_\nu : h \in L_a^2(\nu, H_{0,0}^1) \rightarrow \int_W \langle \xi, h \rangle_{H^1} \, d\nu \in \mathbb{R}.$$

**The involved set of laws of semi-martingales.** We will perform our variations on  $\mathbb{S}$ , the subset of the  $\nu \in \mathbf{M}_1(W)$  such that the evaluation process has a structure of the specific form

$$W_t = W_0 + M_t^\nu + b_t^\nu, \quad \forall t \in [0, 1], \quad \nu - a.s.,$$

where  $(M_t^\nu)$  is a continuous  $\mathbb{R}^d$ -valued  $(\mathcal{F}_t^\nu)$ -local martingale, whose *covariation process*  $\langle (M^\nu)^i, (M^\nu)^j \rangle_t$  is of the specific form

$$\langle (M^\nu)^i, (M^\nu)^j \rangle_t = \int_0^t [\alpha_t^\nu]^{ij} \, dt, \quad \nu - a.s., \quad \forall i, j \in \{1, \dots, d\},$$

where the  $\mathbb{R}^d \otimes \mathbb{R}^d$ -valued stochastic process  $(\alpha_t^\nu)$  is chosen to be predictable, and where  $(b_t^\nu)$  is assumed to be  $(\mathcal{F}_t^\nu)$ -adapted and absolutely continuous, of the form

$$b^\nu := \int_0^\cdot v_s^\nu ds, \quad \nu - a.s..$$

From classical results around the *Lebesgue differentiation theorem* (see [20]), it is easy to see that  $(v_s^\nu)$  can be chosen to be predictable, or merely optional, as it is well known. We call any such  $(v_t^\nu, \alpha_t^\nu)$  the *local characteristics* of  $\nu \in \mathbb{S}$ .

### 3. Associated Semi-martingale Optimization Problems

In this section, we consider measurable cost maps

$$\mathcal{L} : [0, 1] \times \mathbb{R}^d \times \mathbb{R}^d \times (\mathbb{R}^d \otimes \mathbb{R}^d) \rightarrow \mathbb{R} \cup \{+\infty\},$$

which are called subsequently regular Lagrangians, in the acceptance of Definition 5.2. of [33], if  $\mathcal{L}$  satisfies the following assumptions:

- $\text{Dom}\mathcal{L} = [0, 1] \times \mathbb{R}^d \times \mathbb{R}^d \times (\mathbb{R}^d \otimes \mathbb{R}^d)$ , where  $\text{Dom}\mathcal{L} := \{(t, x, v, a) : \mathcal{L} < +\infty\}$ ;
- For all  $(t, x, v, a) \in \text{Dom}\mathcal{L}$ , the function

$$\tilde{\mathcal{L}}(t, x, v, a) : (\tilde{x}, \tilde{v}) \in \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathcal{L}_t(x + \tilde{x}, v + \tilde{v}, a) \in \mathbb{R}$$

is Fréchet differentiable at  $(\mathbf{0}_{\mathbb{R}^d}, \mathbf{0}_{\mathbb{R}^d})$ .

- The mappings  $(t, x, v, a) \in \text{Dom}\mathcal{L} \rightarrow \partial_x \mathcal{L}_t(x, v, a) \in \mathbb{R}^d$  and  $(t, x, v, a) \in \text{Dom}\mathcal{L} \rightarrow \partial_v \mathcal{L}_t(x, v, a) \in \mathbb{R}^d$  are Borel measurable.

$\text{Dom}\mathcal{L}$  notably enlightens the notations. The so-called action functional associated to  $\mathcal{L}$  is defined on  $\mathbb{S}$  by

$$\mathcal{S}(\nu) := \mathbb{E}_\nu \left[ \int_0^1 \mathcal{L}_t(W_t, v_t^\nu, \alpha_t^\nu) dt \right], \quad (3.1)$$

if  $\mathbb{E}_\nu \left[ \int_0^1 |\mathcal{L}_t(W_t, v_t^\nu, \alpha_t^\nu)| dt \right] < +\infty$ , and  $+\infty$  otherwise, for all  $\nu \in \mathbb{S}$ ; we may extend it to a map on  $M_1(W)$  by setting  $\mathcal{S}(\nu) := +\infty$ , for all  $\nu \in \mathbf{M}_1(W) \setminus \mathbb{S}$ . We denote by  $D\mathcal{L}_{t,x,v,a} : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  the map

$$(\tilde{x}, \tilde{v}) \in \mathbb{R}^d \times \mathbb{R}^d \rightarrow \langle (\partial_x \mathcal{L}_t)(x, v, a), \tilde{x} \rangle_{\mathbb{R}^d} + \langle (\partial_v \mathcal{L}_t)(x, v, a), \tilde{v} \rangle_{\mathbb{R}^d} \in \mathbb{R}, \quad (3.2)$$

the differential of  $\tilde{\mathcal{L}}(t, x, v, a)$ , with  $(t, x, v, a) \in \text{Dom}\mathcal{L}$ . Subsequently, given  $\nu_0, \nu_1 \in \mathbf{M}_1(\mathbb{R}^d)$ , the set of transport plans (see [51]) from  $\nu_0$  to  $\nu_1$  is denoted by

$$\Pi(\nu_0, \nu_1) := \{ \gamma \in \mathbf{M}_1(\mathbb{R}^d \times \mathbb{R}^d) : p_{1\star} \gamma = \nu_0, p_{2\star} \gamma = \nu_1 \}, \quad (3.3)$$

$p_1$  and  $p_2$  denoting the canonical projections of  $\mathbb{R}^d \times \mathbb{R}^d$ , and  $\star$  denoting the pushforward of measure. Moreover, we define the function

$$(W_0, W_1) : \omega \in W \rightarrow (\omega(0), \omega(1)) \in \mathbb{R}^d \times \mathbb{R}^d,$$

whose continuity entails the continuity of

$$(W_0, W_1)_\star : \nu \in \mathbf{M}_1(W) \rightarrow (W_0, W_1)_\star \nu \in \mathbf{M}_1(\mathbb{R}^d \times \mathbb{R}^d) \quad (3.4)$$

with respect to the respective topologies of weak convergence in measure (see [7]).

To obtain compact notation in the statement of the following Theorem 3.1, given a non-negative regular Lagrangian  $\mathcal{L}$ , taking  $p_1, p_2 \geq 2$  and a strictly positive continuous function  $f : \mathbb{R}^d \rightarrow \mathbb{R}_+$ , we denote by

$$F_\epsilon^{f, p_1, p_2}(t, x, v, a, \tilde{x}, \tilde{v}) := \frac{|\mathcal{L}_t(x + \epsilon\tilde{x}, v + \epsilon\tilde{v}, a) - \mathcal{L}_t(x, v, a) - \epsilon D\mathcal{L}_{t, x, v, a}[\tilde{x}, \tilde{v}]|}{\epsilon f(\tilde{x}) (1 + \|\tilde{v}\|_{\mathbb{R}^d}^2 + G_{p_1, p_2}(t, x, v, a))},$$

where

$$G_{p_1, p_2}(t, x, v, a) := |\mathcal{L}_t(x, v, a)| + \|\partial_x \mathcal{L}_t(x, v, a)\|_{\mathbb{R}^d}^{p_1} + \|\partial_v \mathcal{L}_t(x, v, a)\|_{\mathbb{R}^d}^{p_2},$$

for all  $(\epsilon, t, x, v, a, \tilde{x}, \tilde{v}) \in \mathbb{R} \times \text{Dom}\mathcal{L} \times \mathbb{R}^d \times \mathbb{R}^d$ . Moreover  $L_{sym+}^1(\lambda, \mathbb{R}^d \otimes \mathbb{R}^d)$  will denote the set of the  $\mathbb{R}^d \otimes \mathbb{R}^d$ -valued Borel measurable functions  $\alpha : t \in [0, 1] \rightarrow \alpha_t \in \mathbb{R}^d \otimes \mathbb{R}^d$  on  $[0, 1]$ , such that  $\alpha_t$  is symmetric non-negative for all  $t \in [0, 1]$ , and

$$\int_0^1 \|\alpha_t\|_{\mathbb{R}^d \otimes \mathbb{R}^d} dt < +\infty.$$

Denoting by  $\mathcal{S}$  the action functional associated to  $\mathcal{L}$  by (3.1), and given  $\nu_0, \nu_1 \in \mathbf{M}_1(\mathbb{R}^d)$  which further satisfy

$$\int_{\mathbb{R}^d} \|x\|_{\mathbb{R}^d} \nu_0(dx) + \int_{\mathbb{R}^d} \|x\|_{\mathbb{R}^d} \nu_1(dx) < +\infty, \quad (3.5)$$

and  $\gamma \in \Pi(\nu_0, \nu_1)$ , Theorem 3.1. is interested in extremal conditions of specific optimization problems over precise subsets of

$$\mathbb{S}_{\nu_0, \nu_1}^1 := \{\nu \in \mathbb{S} : \mathbb{E}_\nu[\|W_t\|_{\mathbb{R}^d}] < +\infty, \forall t \in [0, 1], W_{0*}\nu = \nu_0, W_{1*}\nu = \nu_1\},$$

and of

$$\mathbb{S}_\gamma^1 := \{\nu \in \mathbb{S} : \mathbb{E}_\nu[\|W_t\|_{\mathbb{R}^d}] < +\infty, \forall t \in [0, 1], (W_0, W_1)_*\nu = \gamma\}.$$

To be accurate, once  $l \in H^1$  and  $\alpha \in L_{sym+}^1(\lambda, \mathbb{R}^d \otimes \mathbb{R}^d)$  are given, the latter are determined by

$$I_{l, \alpha}(\gamma) := \inf(\{\mathcal{S}(\nu) : \nu \in \mathbb{A}_\gamma^{l, \alpha}\}), \quad (3.6)$$

where  $\mathbb{A}_\gamma^{l, \alpha}$  denotes the subset

$$\left\{ \nu \in \mathbb{S}_\gamma^1 : \mathbb{E}_\nu[W_t] = \mathbb{E}_\nu[W_0] + l(t), \forall t \in [0, 1], \int_0^\cdot \alpha_t^\nu dt = \int_0^\cdot \alpha_t dt, \nu - a.s. \right\},$$

and by

$$J_{l, \alpha}(\nu_0, \nu_1) := \inf(\{\mathcal{S}(\nu) : \nu \in \mathbb{A}_{\nu_0, \nu_1}^{l, \alpha}\}), \quad (3.7)$$

where  $\mathbb{A}_{\nu_0, \nu_1}^{l, \alpha}$  denotes the subset

$$\left\{ \nu \in \mathbb{S}_{\nu_0, \nu_1}^1 : \mathbb{E}_\nu[W_t] = \mathbb{E}_\nu[W_0] + l(t), \forall t \in [0, 1], \int_0^\cdot \alpha_t^\nu dt = \int_0^\cdot \alpha_t dt, \nu - a.s. \right\}.$$

**Theorem 3.1.** *Given  $l \in H^1$  and  $\alpha \in L_{sym+}^1(\lambda, \mathbb{R}^d \otimes \mathbb{R}^d)$ , let  $\mathcal{L}$  be a non-negative regular Lagrangian, whose associated action  $\mathcal{S}$  is given by (3.1). Further assume that there exists a  $\nu_{opt} \in \mathbb{S}$  which attains the infimum  $J_{l, \alpha}(\nu_0, \nu_1)$  of (3.7) (resp.*



$I_{l,\alpha}(\gamma)$  of (3.6)), and that those infimums are finite, where  $\nu_0, \nu_1 \in \mathbf{M}_1(\mathbb{R}^d)$  satisfy (3.5), and where  $\gamma \in \Pi(\nu_0, \nu_1)$ . If there exists a strictly positive continuous function  $f : \mathbb{R}^d \rightarrow \mathbb{R}_+$ , and  $p_1, p_2 \geq 2$ , which satisfy both conditions

$$\limsup_{\epsilon \rightarrow 0} \sup_{(t,x,v,a,\tilde{x},\tilde{v}) \in \text{Dom}\mathcal{L} \times \mathbb{R}^d \times \mathbb{R}^d} F_\epsilon^{f,p_1,p_2}(t,x,v,a,\tilde{x},\tilde{v}) = 0, \quad (3.8)$$

and

$$\sup_{(t,x,v,a) \in \text{Dom}\mathcal{L}} \frac{\|\partial_x \mathcal{L}_t(x,v,a)\|_{\mathbb{R}^d}^{p_1} + \|\partial_v \mathcal{L}_t(x,v,a)\|_{\mathbb{R}^d}^{p_2}}{1 + \mathcal{L}_t(x,v,a)} < +\infty, \quad (3.9)$$

then there exists a càd-làg  $(\mathcal{F}_t^{\nu_{opt}})$ -martingale  $(N_t^{\nu_{opt}})_{t \in [0,1]}$  on  $(W, \mathcal{B}_W^{\nu_{opt}}, \nu_{opt})$ , and a measurable deterministic process  $(A_t^{\nu_{opt}})$ , such that

$$\partial_v \mathcal{L}_t(W_t, v_t^{\nu_{opt}}, \alpha_t^{\nu_{opt}}) - \int_0^t \partial_x \mathcal{L}_s(W_s, v_s^{\nu_{opt}}, \alpha_s^{\nu_{opt}}) ds = A_t^{\nu_{opt}} + N_t^{\nu_{opt}}, \lambda \otimes \nu_{opt} - a.e.. \quad (3.10)$$

Moreover, under the same conditions, the result still holds with sup instead of inf in (3.6) resp. in (3.7).

*Proof.* Assume that  $I_{l,\alpha}(\gamma) < +\infty$ , and that  $\nu_{opt}$  attains the associated optimum. We obtain  $\mathcal{S}(\nu_{opt}) < +\infty$  and

$$\mathbb{E}_{\nu_{opt}} \left[ \int_0^1 (\|\partial_x \mathcal{L}_s(W_s, v_s^{\nu_{opt}}, \alpha_s^{\nu_{opt}})\|_{\mathbb{R}^d}^{p_1} + \|\partial_v \mathcal{L}_s(W_s, v_s^{\nu_{opt}}, \alpha_s^{\nu_{opt}})\|_{\mathbb{R}^d}^{p_2}) ds \right] < +\infty. \quad (3.11)$$

Thus, from Theorem 5.1. of [33],  $\mathcal{S}$  is  $L_a^2(\nu, H_{0,0}^1)$ -differentiable at  $\nu_{opt}$ . Taking  $h \in A_{\nu_{opt}}^{0,\infty}$ , since  $h \in V_{\nu_{opt}}^{0,\infty}$ , we obtain

$$\delta \mathcal{S}_{\nu_{opt}}[h] = \frac{d}{d\epsilon} \mathcal{S}(\tau_{\epsilon h \star} \nu_{opt}) \Big|_{\epsilon=0}, \quad (3.12)$$

where  $\tau_{\epsilon h} := I_W + \epsilon h$ ,  $I_W$  denoting the identity on  $W$ . To enlighten the notation we set  $\nu_{\epsilon h} := \tau_{\epsilon h \star} \nu_{opt}$ . We have

$$\int_0^\cdot \alpha_s^{\nu_{\epsilon h}} \circ \tau_{\epsilon h} ds = \int_0^\cdot \alpha_s^{\nu_{opt}} ds = \int_0^\cdot \alpha_s ds, \nu_{opt} - a.s.,$$

so that

$$\int_0^\cdot \alpha_s^{\nu_{\epsilon h}} ds = \int_0^\cdot \alpha_s ds, \nu_{\epsilon h} - a.s..$$

Since

$$h_0 = h_1 = \mathbf{0}_{\mathbb{R}^d}, \nu_{opt} - a.s.,$$

we obtain that  $(W_0, W_1)_{\star} \nu_{\epsilon h} = \gamma$ , and similarly that  $W_{0\star} \nu_{\epsilon h} = \nu_0$ , and  $W_{1\star} \nu_{\epsilon h} = \nu_1$ , with  $\epsilon \in \mathbb{R}$ . Moreover

$$\mathbb{E}_{\nu_{\epsilon h}}[W_t] = \mathbb{E}_{\nu_{opt}}[W_t] + \epsilon \mathbb{E}_{\nu_{opt}}[h_t] = \mathbb{E}_{\nu_{opt}}[W_0] + l(t) = \mathbb{E}_{\nu_{\epsilon h}}[W_0] + l(t),$$

for all  $t \in [0,1]$ , since  $\int_W h d\nu_{opt} = \mathbf{0}_{H^1}$ , and  $W_{0\star} \nu_{opt} = W_{0\star} \nu_{\epsilon h}$ . Thus, for all  $\epsilon \in \mathbb{R}$ , in both cases we obtain

$$\mathcal{S}(\nu_{\epsilon h}) - \mathcal{S}(\nu_{opt}) \geq 0.$$

Together with (3.12), it yields

$$\delta \mathcal{S}_{\nu_{opt}}[h] \geq 0.$$

Since this also holds with  $-h$ , by linearity we obtain  $\delta\mathcal{S}_{\nu_{opt}}[h] = 0$ , for all  $h \in A_{\nu_{opt}}^{0,\infty}$ . By a result of [34] which determines the closure of  $A_{\nu_{opt}}^{0,\infty}$ , and by continuity of  $\delta\mathcal{S}_{\nu_{opt}} : L_a^2(\nu, H_{0,0}^1) \rightarrow \mathbb{R}$ , it yields  $\delta\mathcal{S}_{\nu_{opt}}[h] = 0$ , for all  $h \in L_a^2(\nu_{opt}, H_{0,0}^1)$  such that  $\int_W h d\nu_{opt} = \mathbf{0}_{H^1}$ . Hence, by applying the main theorem of [34], we obtain the result of  $I_{l,\alpha}(\gamma)$ . Similarly, if  $J_{l,\alpha}(\nu_0, \nu_1) < +\infty$  and  $\nu_{opt}$  attains its optimum, by applying the previous result to  $\gamma := (W_0, W_1)_* \nu_{opt} \in \Pi(\nu_0, \nu_1)$ , the result follows. With a sup instead of inf in (3.6) resp. in (3.7), the proof is similar.  $\square$

#### 4. An Entropic Approach on the Classical Wiener Space

Given  $\eta, \nu \in \mathbf{M}_1(W)$ , recall that the relative entropy of  $\nu$  with respect to  $\eta$  is defined by

$$\mathcal{H}(\nu|\eta) = \begin{cases} \mathbb{E}_\eta \left[ \frac{d\nu}{d\eta} \ln \frac{d\nu}{d\eta} \right], & \text{if } \nu \ll \eta \text{ (i.e. absolutely continuous)}; \\ +\infty, & \text{otherwise} \end{cases}; \quad (4.1)$$

we refer to [17] for a synthetic recall on this function's main properties. In this section, the reference measures will be specific probability measures which are absolutely continuous with respect to Wiener measures (see [30], [52]). Given  $\nu_0 \in \mathbf{M}_1(\mathbb{R}^d)$ , we denote by  $\mu_{\nu_0}$  the Wiener measure with initial distribution  $\nu_0$ . Let  $\mathcal{V} : \mathbb{R}^d \rightarrow [0, +\infty)$  be a non-negative smooth map such that

$$\mathbb{E}_{\mu_{\nu_0}} \left[ \int_0^1 |\mathcal{V}(W_s)| ds \right] < +\infty. \quad (4.2)$$

We define  $Z_{\nu_0} : \mathbb{R}^d \rightarrow \mathbb{R}$  to be a  $\mathcal{B}_{\mathbb{R}^d}^{\nu_0} / \mathcal{B}_{\mathbb{R}}$ -measurable map such that

$$Z_{\nu_0} \circ W_0 = \mathbb{E}_{\mu_{\nu_0}} \left[ \exp \left( - \int_0^1 \mathcal{V}(W_s) ds \right) \middle| \sigma(W_0) \right], \quad \mu_{\nu_0} - a.s.,$$

and we define  $\mu_{\mathcal{V}}^{\nu_0}$  to be the probability measure, absolutely continuous with respect to the Wiener measure  $\mu_{\nu_0}$ , with Radon-Nikodym derivative

$$\frac{d\mu_{\mathcal{V}}^{\nu_0}}{d\mu_{\nu_0}} := \frac{\exp \left( - \int_0^1 \mathcal{V}(W_s) ds \right)}{Z_{\nu_0}(W_0)}, \quad \mu_{\nu_0} - a.s.,$$

so that  $W_{0*} \mu_{\mathcal{V}}^{\nu_0} = \nu_0$ . Recall that, when we have  $\mathcal{H}(\nu|\mu_{\mathcal{V}}^{\nu_0}) < +\infty$ , then we also have

$$\mathcal{H}(\nu|\mu_{\mathcal{V}}^{\nu_0}) = \mathcal{H}(\nu|\mu_{\nu_0}) + \mathbb{E}_\nu \left[ \int_0^1 \mathcal{V}(W_s) ds \right] + \ln(\mathcal{Z}_{\nu_0}), \quad (4.3)$$

where

$$\mathcal{Z}_{\nu_0} := \exp(\mathbb{E}_{\nu_0}[\ln Z_{\nu_0}]). \quad (4.4)$$

The following Lemma follows from classical techniques.

**Lemma 4.1.** *Let  $\nu_0, \nu_1 \in \mathbf{M}_1(\mathbb{R}^d)$  which satisfy (3.5), and  $\gamma \in \Pi(\nu_0, \nu_1)$  (see (3.3)),  $l \in H^1$ . Define*

$$m_{\nu_0, \nu_1}^l := \inf \left( \left\{ \mathcal{H}(\nu|\mu_{\mathcal{V}}^{\nu_0}) : \nu \in M_1^{l; \nu_0, \nu_1}(W) \right\} \right), \quad (4.5)$$

where

$$M_1^{l; \nu_0, \nu_1}(W) := \{ \nu \in M_1^{\nu_0, \nu_1}(W) : \mathbb{E}_\nu[W_t] = \mathbb{E}_{\nu_0}[W_0] + l(t), \quad \forall t \in [0, 1] \},$$

and where  $M_1^{\nu_0, \nu_1}(W)$  denotes the set

$$\{\nu \in \mathbf{M}_1(W) : \mathbb{E}_\nu[\|W_t\|_{\mathbb{R}^d}] < +\infty, \forall t \in [0, 1], W_{0*}\nu = \nu_0, W_{1*}\nu = \nu_1\}.$$

Moreover, define

$$\tilde{m}_\gamma^l := \inf \left( \left\{ \mathcal{H}(\nu | \mu_{\mathcal{V}}^{\nu_0}) : \nu \in M_1^{l; \gamma}(W) \right\} \right), \quad (4.6)$$

where  $M_1^{l; \gamma}(W)$  denotes

$$\{\nu \in M_1^\gamma(W) : \mathbb{E}_\nu[W_t] = \mathbb{E}_\nu[W_0] + l(t), \forall t \in [0, 1]\},$$

and where

$$M_1^\gamma(W) := \{\nu \in \mathbf{M}_1(W) : \mathbb{E}_\nu[\|W_t\|_{\mathbb{R}^d}] < +\infty, \forall t \in [0, 1], (W_0, W_1)_*\nu = \gamma\}.$$

By further assuming that  $m_{\nu_0, \nu_1}^l < +\infty$ , (resp. that  $\tilde{m}_\gamma^l < +\infty$ ), then there exists a unique  $\nu_{opt} \in \mathbf{M}_1(W)$  which attains the infimum of (4.5) (resp. of (4.6)).

*Proof.* Assume that  $\tilde{m}_\gamma^l < +\infty$ , and let  $(\nu_n)_{n \in \mathbb{N}}$  be a minimizing sequence such that  $(\mathcal{H}(\nu_n | \mu_{\mathcal{V}}^{\nu_0}))_{n \in \mathbb{N}}$  converges to  $\tilde{m}_\gamma^l$ , which further satisfies

$$\mathcal{H}(\nu_n | \mu_{\mathcal{V}}^{\nu_0}) \leq \tilde{m}_\gamma^l + 1; \quad (4.7)$$

in particular  $\nu_n \in M_1^{l; \gamma}(W)$  for any  $n \in \mathbb{N}$ . From the Poussin–La Vallée criterion (see [15]), we see that  $(\frac{d\nu_{\sigma(n)}}{d\mu_{\mathcal{V}}^{\nu_0}})_{n \in \mathbb{N}}$  is uniformly integrable. Hence, the Dunford–Pettis theorem (see [15]) ensures that it is relatively compact in the weak topology  $\sigma(L^1, L^\infty)$  of  $L^1(\mu_{\mathcal{V}}^{\nu_0})$ . Further, we extract a subsequence  $(\frac{d\nu_{\sigma(n)}}{d\mu_{\mathcal{V}}^{\nu_0}})_{n \in \mathbb{N}}$  which converges to a certain  $L \in L^1(\mu_{\mathcal{V}}^{\nu_0})$  weakly in  $L^1(\mu_{\mathcal{V}}^{\nu_0})$ ; see [10]. Taking indicator  $\mathbf{1}_A$ ,  $A \in \mathcal{B}_W$ , it is an easy task to check that  $L \geq 0$   $\mu_{\mathcal{V}}^{\nu_0}$ -a.s., and taking  $A = W$ , that  $\mathbb{E}_{\mu_{\mathcal{V}}^{\nu_0}}[L] = 1$ . Thus, we can define a Borel probability measure  $\nu_{opt}$ , to be the probability absolutely continuous with respect to  $\mu_{\mathcal{V}}^{\nu_0}$  with Radon–Nikodym derivative  $L$ . The rest of the proof amounts to prove that  $\nu_{opt}$  is the unique minimum of (4.6). Taking  $f \in C_b(W)$ , still from the weak  $L^1(\mu_{\mathcal{V}}^{\nu_0})$  convergence, we see that  $(\nu_{\sigma(n)})_{n \in \mathbb{N}}$  converges to  $\nu_{opt}$  in the topology of weak convergence in measure. In particular, since the continuity of  $W_0$  entails the continuity of  $W_{0*}$ , we obtain  $W_{0*}\nu_{opt} = \nu_0$ . Since the entropy is lower semi-continuous, and  $(\nu_{\sigma(n)})$  is a minimizing sequence, we obtain

$$\mathcal{H}(\nu_{opt} | \mu_{\mathcal{V}}^{\nu_0}) \leq \liminf \mathcal{H}(\nu_{\sigma(n)} | \mu_{\mathcal{V}}^{\nu_0}) \leq \tilde{m}_\gamma^l < \tilde{m}_\gamma^l + 1.$$

Hence, to prove that  $\nu_{opt}$  attains the infimum, it is enough to prove that  $\nu_{opt} \in M_1^{l; \gamma}(W)$ . Let  $K := (\cup_{n \in \mathbb{N}} \{\nu_{\sigma(n)}\}) \cup \{\nu_{opt}\}$ . Since  $\mathcal{V} \geq 0$ , from (4.3), we have

$$\mathcal{H}(\nu | \mu_{\nu_0}) \leq \mathcal{H}(\nu | \mu_{\mathcal{V}}^{\nu_0}) - \ln(\mathcal{Z}_{\nu_0}) \leq \tilde{m}_\gamma^l + 1 - \ln(\mathcal{Z}_{\nu_0}),$$

for all  $\nu \in K$ . In particular, any  $\nu \in K$  is absolutely continuous with respect to  $\mu_{\nu_0}$ . Whence, from the Girsanov theorem (see [16], [21], and [22]), there exists a  $(\mathcal{F}_t^\nu)$ -Brownian motion  $(B_t^\nu)$  on  $(W, \mathcal{B}_W^\nu, \nu)$ , and  $b^\nu := \int_0^\cdot v_s^\nu ds \in L_a^2(\nu, H^1)$ , where  $(v_t^\nu)$  is chosen to be  $(\mathcal{F}_t^\nu)$ -predictable, such that

$$W_t - W_0 = B_t^\nu + \int_0^t v_s^\nu ds, \forall t \in [0, 1], \nu - a.s..$$

Moreover, since

$$W_{0*}\nu = \nu_0 = W_{0*}\mu_{\nu_0},$$

from the entropy representation formula of [21], which stands on developments of the Girsanov theorem (see [22]), providing precise representations of the Radon-Nikodym derivatives with respect to the classical Wiener measure (for instance see [18], [19], [28]), which involve sharp Itô's stochastic integrals ([26],[40]), we obtain

$$2\mathcal{H}(\nu|\mu_{\nu_0}) = \mathbb{E}_\nu \left[ \int_0^1 \|v_t^\nu\|_{\mathbb{R}^d}^2 dt \right].$$

Whence, by applying the Cauchy-Schwarz inequality together with Jensen's inequality,

$$\sup_{\nu \in K} \mathcal{H}(\nu|\mu_{\nu_0}) \leq \tilde{m}_\gamma^l + 1 - \ln(\mathcal{Z}_{\nu_0}) < +\infty$$

implies

$$c := \sup_{\nu \in K} \mathbb{E}_\nu [\|W_t - W_0\|_{\mathbb{R}^d}^2] < +\infty, \quad (4.8)$$

for all  $t \in [0, 1]$ . In particular, (3.5) and (4.8) yield  $E_{\nu_{opt}} [\|W_t\|_{\mathbb{R}^d}] < +\infty$ , for all  $t \in [0, 1]$ . On the other hand, from the Cauchy-Schwarz inequality, together with the Bienaymé-Chebyshev inequality (see [47]), we obtain

$$\left\| \mathbb{E}_\nu \left[ (W_t - W_0) \mathbf{1}_{\|W_t - W_0\|_{\mathbb{R}^d} > N} \right] \right\|_{\mathbb{R}^d} \leq \frac{\mathbb{E}_\nu [\|W_t - W_0\|_{\mathbb{R}^d}^2]}{N}, \quad (4.9)$$

for all  $N \in \mathbb{N}^*$ ,  $t \in [0, 1]$ . Given  $\epsilon > 0$ , and  $t \in [0, 1]$ , together with (4.8), (4.9) yields the existence of  $N_\epsilon \in \mathbb{N}^*$  such that

$$\sup_{\nu \in K} \left\| \mathbb{E}_\nu \left[ (W_t - W_0) \mathbf{1}_{\|W_t - W_0\|_{\mathbb{R}^d} > N_\epsilon} \right] \right\|_{\mathbb{R}^d} \leq \frac{\epsilon}{2}. \quad (4.10)$$

Since  $\nu_{\sigma(n)} \in M_1^{l,\gamma}(W)$ , for all  $n \in \mathbb{N}$ , the triangular inequality yields, for any  $t \in [0, 1]$ ,  $n \in \mathbb{N}$

$$\|\mathbb{E}_{\nu_{opt}} [W_t] - \mathbb{E}_{\nu_{opt}} [W_0] - l(t)\|_{\mathbb{R}^d} \leq A_\epsilon(n) + B_\epsilon(n), \quad (4.11)$$

where  $A_\epsilon(n)$  denotes

$$\left\| \mathbb{E}_{\nu_{opt}} \left[ (W_t - W_0) \mathbf{1}_{\|W_t - W_0\|_{\mathbb{R}^d} \leq N_\epsilon} \right] - \mathbb{E}_{\nu_{\sigma(n)}} \left[ (W_t - W_0) \mathbf{1}_{\|W_t - W_0\|_{\mathbb{R}^d} \leq N_\epsilon} \right] \right\|_{\mathbb{R}^d},$$

and where  $B_\epsilon(n)$  denotes

$$\left\| \mathbb{E}_{\nu_{opt}} \left[ (W_t - W_0) \mathbf{1}_{\|W_t - W_0\|_{\mathbb{R}^d} > N_\epsilon} \right] - \mathbb{E}_{\nu_{\sigma(n)}} \left[ (W_t - W_0) \mathbf{1}_{\|W_t - W_0\|_{\mathbb{R}^d} > N_\epsilon} \right] \right\|_{\mathbb{R}^d}.$$

Using (4.10), we obtain  $B_\epsilon(n) \leq \epsilon$ . Thus, from (4.11), we obtain

$$\|\mathbb{E}_{\nu_{opt}} [W_t] - \mathbb{E}_{\nu_{opt}} [W_0] - l(t)\|_{\mathbb{R}^d} \leq \epsilon + A_\epsilon(n). \quad (4.12)$$

Moreover, since  $(W_t^i - W_0^i) \mathbf{1}_{\|W_t - W_0\|_{\mathbb{R}^d} \leq N_\epsilon} \in L^\infty(\mu_{\nu_0}^i)$  for all  $i \in \{1, \dots, d\}$ , the  $\sigma(L^1, L^\infty)$  convergence of  $(\frac{d\nu_{\sigma(n)}}{d\mu_{\nu_0}^i})_{n \in \mathbb{N}}$  to  $\frac{d\nu_{opt}}{d\mu_{\nu_0}^i}$  yields

$$\lim_{n \rightarrow \infty} A_\epsilon(n) = 0.$$

Whence, (4.12) yields

$$\|\mathbb{E}_{\nu_{opt}} [W_t] - \mathbb{E}_{\nu_{opt}} [W_0] - l(t)\|_{\mathbb{R}^d} \leq \epsilon.$$

Since this holds for all  $\epsilon > 0$ , we conclude that

$$\mathbb{E}_{\nu_{opt}} [W_t] = \mathbb{E}_{\nu_{opt}} [W_0] + l(t),$$

for all  $t \in [0, 1]$ . Moreover, since  $(\nu_{\sigma(n)})$  converges weakly in law to  $\nu_{opt}$ , and  $\nu_{\sigma(n)} \in M_1^{l;\gamma}(W)$  for all  $n \in \mathbb{N}$ , the continuity of  $(W_0, W_1)_*$  (see (3.4)) yields

$$(W_0, W_1)_* \nu_{opt} = \gamma.$$

Finally, we have  $\nu_{opt} \in M_1^{l;\gamma}(W)$ , so that it is a minimum of (4.6). The uniqueness follows from the strict convexity of the relative entropy on the convex set  $M_1^{l;\gamma}(W)$ . Moreover, from the continuity of  $W_{0*}$  and of  $W_{1*}$ , the corresponding result of (4.5) follows similarly.  $\square$

*Remark 4.2.* From arguments similar to those used to obtain (4.8), given  $\nu \in \mathbf{M}_1(W)$  such that  $\nu_0 := W_{0*}\nu \in \mathbf{M}_1(\mathbb{R}^d)$  satisfies  $\int_{\mathbb{R}^d} \|x\|_{\mathbb{R}^d} \nu_0(dx) < +\infty$ , and  $\mathcal{H}(\nu|\mu_{\mathcal{V}}^{\nu_0}) < +\infty$ , note that the condition  $\mathbb{E}_{\nu} [\|W_t\|_{\mathbb{R}^d}] < +\infty, \forall t \in [0, 1]$ , easily follows.

In Theorem 4.3 below, we use specific Lagrangians with a usual convention of *E.Q.M.* (see [36], [54]).

**Theorem 4.3.** *Let  $\nu_0, \nu_1 \in \mathbf{M}_1(\mathbb{R}^d)$ , be such that (3.5) holds,  $\gamma \in \Pi(\nu_0, \nu_1)$  (see (3.3)),  $l \in H^1$ , and let  $\mathcal{V} : \mathbb{R}^d \rightarrow [0, +\infty)$  be a non negative smooth map which satisfies condition (4.2). Further assume that the map  $\mathcal{L}^{\mathcal{V}}$ , which is defined by*

$$\mathcal{L}_t^{\mathcal{V}}(x, v, a) := \frac{\|v\|_{\mathbb{R}^d}^2}{2} + \mathcal{V}(x),$$

for all  $(t, x, v, a) \in [0, 1] \times \mathbb{R}^d \times \mathbb{R}^d \times (\mathbb{R}^d \otimes \mathbb{R}^d)$ , meets the hypothesis of Theorem 3.1, and that  $I_{l, I_{\mathbb{R}^d}}(\gamma) < +\infty$  (resp. that  $J_{l, I_{\mathbb{R}^d}}(\nu_0, \nu_1) < +\infty$ ); see Theorem 3.1. Then, there exists a unique probability  $\nu_{opt} \in \mathbb{S}$ , which attains the infimum (3.6) (resp. (3.7)), with the Lagrangian  $\mathcal{L}^{\mathcal{V}}$ , and  $\alpha_t = I_{\mathbb{R}^d}, \forall t \in [0, 1]$ . Moreover,  $\nu_{opt}$  is absolutely continuous with respect to the Wiener measure  $\mu_{\nu_0}$ , and satisfies the average preserving Euler–Lagrange condition (3.10).

*Proof.* Let  $\nu$  be an element of the subset

$$S_l := \left\{ \nu \in \mathbb{S}_{\gamma}^1 : \int_0^1 \alpha_t^{\nu} dt = \int_0^1 \alpha_t dt, \nu - a.s. \right\}.$$

Since  $\alpha_t = I_{\mathbb{R}^d}$ , for all  $t \in [0, 1]$ , and  $\nu \in \mathbb{S}$ , Levy’s criterion (see [25]) ensures that the martingale part of  $\nu$  is a  $(\mathcal{F}_t^{\nu})$ –Brownian motion on  $(W, \mathcal{B}_W^{\nu}, \nu)$ . Thus, we have

$$W_t = W_0 + B_t^{\nu} + \int_0^t v_s^{\nu} ds, \forall t \in [0, 1], \nu - a.s.,$$

for some  $(\mathcal{F}_t^{\nu})$ –Brownian motion on  $(W, \mathcal{B}_W^{\nu}, \nu)$ . Since  $I_{l, I_{\mathbb{R}^d}}(\gamma) < +\infty$ , for  $\nu \in S_l$  such that

$$\mathbb{E}_{\nu} \left[ \int_0^1 |\mathcal{L}_t^{\mathcal{V}}(W_t, v_t^{\nu}, \alpha_t^{\nu})| dt \right] < +\infty,$$

by a classical application of the Girsanov theorem (see [22], [29], [50]), we have  $\nu \ll \mu_{\nu_0}$  (absolutely continuous) and from [21] and (4.3), we obtain

$$\mathcal{H}(\nu|\mu_{\nu_0}^{\nu_0}) = \mathcal{S}(\nu) + \ln(\mathcal{Z}_{\nu_0}),$$

where

$$\mathcal{S}(\nu) := \mathbb{E}_{\nu} \left[ \int_0^1 \mathcal{L}_s^{\nu}(W_s, v_s^{\nu}, \alpha_s^{\nu}) ds \right].$$

Conversely if the entropy is finite (see [21]), we have  $\nu \in \mathbb{S}$ , and still by Levy's criterion  $\alpha_t = I_{\mathbb{R}^d}$ ,  $\lambda - a.e.$ . Thus  $\nu_{opt}$  attains the infimum (3.6), if and only if it attains the infimum in (4.6). Hence, the result easily follows by applying Lemma 4.1 and Theorem 3.1. Similarly, the result of (3.7) follows.  $\square$

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RÉMI LASSALLE: UNIVERSITÉ PARIS-DAUPHINE, PSL, PLACE DU MARÉCHAL DE LATTRE DE TASSIGNY, 75775 PARIS CEDEX 16, FRANCE

*E-mail address:* lassalle@ceremade.dauphine.fr