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THE SEMIMARTINGALE DYNAMICS AND GENERATOR OF A CONTINUOUS TIME SEMI-MARKOV CHAIN

ROBERT J. ELLIOTT*

Abstract. We consider a finite state, continuous time homogeneous semi-Markov chain $X = \{X_t, t \geq 0\}$. Without loss of generality the state space of the chain can be identified with the set of unit vectors $S = \{e_1, e_2, \ldots, e_N\}$ where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)' \in R^N$. The probabilistic and dynamic properties of X can be described by either a rate matrix A or a matrix which gives the occupation times in the various states together with the probabilities of jumping to a different state. For a continuous time Markov chain the occupation times are memoryless, implying the distributions are exponential. For semi-Markov chains the occupation times can have more general distributions. The relation between these two descriptions is first investigated and the semimartingale dynamics of a semi-Markov chain obtained in contrast to the traditional description of a semi-Markov chain in terms of a renewal process.

An equation giving the dynamics of the occupation times is derived together with an equation for the density of the conditional occupation time and state. Some approximations for these dynamics are then obtained.

1. Introduction

This paper derives new results and representations for a continuous time, finite state, homogeneous semi-Markov chain $X = \{X_t, t \geq 0\}$. Without loss of generality the state space of the chain can be taken to be the set $S = \{e_1, e_2, \ldots, e_N\}$ of unit vectors in R^N.

For a continuous time Markov chain the amount of time the chain spends in any state has an exponential distribution. This condition is relaxed for a semi-Markov chain: these occupation times may have more general distributions.

Discrete time Markov chains have occupation times which have a geometric distribution, a condition which is relaxed for discrete time semi-Markov chains as discussed in our recent paper [3]. Semi-Markov chains are discussed in [5]. This paper provides the semimartingale dynamics of a semi-Markov chain in contrast to the description in terms of a renewal process.

In many real world applications it appears that semi-Markov processes might be more appropriate models. However, their analysis is more technical and involved. We hope this paper contributes to their analysis.

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In Section 2 we first relate the dynamics and probabilistic properties of a finite state Markov chain and semi-Markov chain when they are described by a rate matrix or by a matrix giving the distribution of occupation times in the various states.

Dynamics for the occupation times are obtained in Section 3. In Section 4 the equation for the density of the occupation time is derived.

2. Dynamics

In this section we compare the dynamics of a continuous time Markov chain with those of a continuous time semi-Markov chain.

Our processes are defined on a probability space $(\Omega, \mathcal{F}, P)$ and time $t \in [0, \infty)$. The state space is identified with a set of unit vectors $S = \{e_1, e_2, \dots, e_N\}$,

$$e_i = (0, 0, \dots, 1, 0, \dots, 0)' \in R^N.$$

2.1. Markov chain. We first consider a continuous time homogeneous Markov chain $X = \{X_t, \, t \geq 0\}$ with $X_t \in S$.

Write

$$p_t = E[X_t | X_0] = (p_t^1, p_t^2, \dots, p_t^N)' \in R^N.$$

The probabilistic properties of X are usually described by a 'rate' matrix $A = (a_{ji}, \, 1 \leq i, \, j \leq N)$ such that

$$\frac{dp_t}{dt} = Ap_t,$$

with $p_0 = X_0$ given. Then $p_t = E[X_t|X_0] = e^{At} X_0 \in R^N$ and more generally for $0 \leq s \leq t$

$$E[X_t | X_s] = e^{A(t-s)} X_s \, .$$

Define the vector process $M := \{M_t, \, t \geq 0\}$ by

$$M_t = X_t - X_0 - \int_0^t AX_u du \in R^N.$$

With $\mathcal{F}_t := \sigma\{X_u : u \leq t\}$ and $0 \leq s \leq t$

$$\begin{aligned}
E[M_t - M_s | \mathcal{F}_s] &= E\big[X_t - X_s - \int_s^t AX_u du | X_s\big] \\
&= e^{A(t-s)} X_s - X_s - \int_s^t AE[X_u|X_s] du \\
&= e^{A(t-s)} X_s - X_s - \int_s^t Ae^{A(u-s)} X_s du \\
&= 0 \in R^N
\end{aligned}$$

so M is an $\{\mathcal{F}_t\}$ vector martingale. Consequently the semimartingale representation of X is, as in [4],

$$X_t = X_0 + \int_0^t AX_u du + M_t \in R^N. \tag{2.1}$$

For $i \neq j$, $a_{ji} \geq 0$ is the rate at which X jumps from e_i to e_j . Also,

$$a_{ii} < 0 \quad \text{and} \quad \sum_{j=1}^{N} a_{ji} = 0.$$

The occupation times which describe how long the Markov chain X remains in any state are exponentially distributed. See [1]. Write

$$0 < \tau_1 < \tau_2 < \dots$$

for the jump times of X. If the occupation time in state e_i is exponentially distributed with parameter λ_i then

$$P(\tau_n < \tau_{n+1} \leq t | X_{\tau_n} = e_i) = 1 - e^{-\lambda_i(t-\tau_n)}. \tag{2.2}$$

If, for $j \neq i$, the probability of jumping from e_i to e_j, given there is a jump,is p_{ji} then

$$P(X_{\tau_{n+1}} = e_j,\ \tau_n \leq \tau_{n+1} \leq t | X_{\tau_n} = e_i) = p_{ji}\big(1 - e^{-\lambda_i(t-\tau_n)}\big). \tag{2.3}$$

Also

$$\sum_{\substack{j=1\\ j\neq i}}^{N} p_{ji} = 1.$$

The rate of jumping from e_i to e_j at time t, given $t \leq \tau_{n+1}$ and $X_{\tau_n} = e_i$, is then $p_{ji}\lambda_i$.

For the Markov chain X this rate is memoryless, that is, it does not depend on τ_n or t.

However, from the previous parameterization of X the rate of jumping from e_i to e_j is a_{ji} , $i \neq j$. Therefore, for $i \neq j$

$$a_{ji} = p_{ji}\lambda_i\ .$$

Now the p_{ji} only refer to non-zero jumps, that is $i \neq j$, and

$$\sum_{\substack{j=1\\ j\neq i}}^{N} p_{ji} = 1.$$

So

$$\sum_{\substack{j=1\\ j\neq i}}^{N} p_{ji}\lambda_i = \lambda_i = \sum_{\substack{j=1\\ j\neq i}}^{N} a_{ji} = -a_{ii}.$$

λ_i is the rate at which the chain jumps from e_i , (to any other state e_j , $i \neq j$).

The distribution describing how long the chain stays in state e_i and does not jump is

$$P(\tau_n \leq t < \tau_{n+1} | X_{\tau_n} = e_i) = e^{-\lambda_i(t-\tau_n)}.$$

Suppose $N = 3$. We have shown that

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & {}_{33} \end{pmatrix} = \begin{pmatrix} -\lambda_1 & p_{12}\lambda_2 & p_{13}\lambda_3 \\ p_{21}\lambda_1 & -\lambda_2 & p_{23}\lambda_3 \\ p_{31}\lambda_1 & p_{32}\lambda_2 & -\lambda_3 \end{pmatrix}\ .$$

With

$$\Pi := \begin{pmatrix} -1 & p_{12} & p_{13} \\ p_{21} & -1 & p_{23} \\ p_{31} & p_{32} & -1 \end{pmatrix}$$

$$D := \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

we see

$$A = \Pi D. \tag{2.4}$$

2.2. Semi-Markov chain. For a semi-Markov chain $X = \{X_t,\, t \geq 0\}$ the occupation times in any state e_i have more general distributions. Suppose for $1 \leq i \leq N$ there is a probability distribution G_i such that

$$P(\tau_n \leq t < \tau_{n+1} | X_{\tau_n} = e_i) = G_i(t - \tau_n).$$

Note that $G_i(t-\tau_n)$ is decreasing in t, or at least non-increasing, so assuming G_i is differentiable with derivative g_i , $g_i(t-\tau_n) \leq 0$. Then

$$P(\tau_n < \tau_{n+1} \leq t | X_{\tau_n} = e_i) = 1 - G_i(t - \tau_n).$$

Suppose

$$p_{ji} = P(X_{\tau_{n+1}} = e_j | X_{\tau_n} = e_i) \quad \text{for} \quad i \neq j.$$

We consider only non-zero jumps so

$$\sum_{\substack{j=1 \\ j \neq i}}^{N} p_{ji} = 1.$$

Then for $i \neq j$:

$$P(X_{\tau_{n+1}} = e_j \,,\, \tau_n < \tau_{n+1} \leq t | X_{\tau_n} = e_i) = p_{ji}\big(1 - G_i(t - \tau_n)\big). \tag{2.5}$$

We consider the analog of the rate matrix $A = (a_{ji}\,,\, 1 \leq i,\, j \leq N)$ for this semi-Markov chain. From (2.5) the rate of jumping from $e_i = X_{\tau_n}$ to $e_j = X_{\tau_{n+1}}$, $i \neq j$, given $X_t = X_{\tau_n} = e_i$ and $\tau_{n+1} > t$ is

$$\frac{\frac{-d}{dt}\big(p_{ji} G_i(t-\tau_n)\big)}{G_i(t-\tau_n)} = -\frac{p_{ji} g_i(t-\tau_n)}{G_i(t-\tau_n)}\,.$$

That is, the off-diagonal elements of a rate matrix for a semi-Markov chain have the form

$$a_{ji}(t-\tau_n) = \frac{-p_{ji} g_i(t-\tau_n)}{G_i(t-\tau_n)}\,.$$

Now

$$\sum_{\substack{j=1 \\ j \neq i}}^{N} p_{ji} = 1$$

so, as in the Markov case,

$$\begin{aligned} a_{ii}(t-\tau_n) &= \sum_{\substack{j=1 \\ i \neq j}}^{N} p_{ji} \, \frac{g_i(t-\tau_n)}{G_i(t-\tau_n)} \\ &= \frac{g_i(t-\tau_n)}{G_i(t-\tau_n)} \, . \end{aligned} \tag{2.6}$$

This is the infinitesimal rate of there not being a jump at t, given $X_t = e_i$ and $\tau_{n+1} > t$.

The matrix $A(t-\tau_n) = \big(a_{ji}(t-\tau_n),\, 1 \leq i,\, j \leq N\big)$ is the analog for the semi-Markov chain of the rate matrix for the Markov chain. Unlike the Markov case the entries now depend on $(t-\tau_n)$, the time since the last jump.

The semimartingale dynamics for the semi-Markov chain are

$$X_t = X_{\tau_n} + \int_{\tau_n}^{\tau_{n+1} \wedge t} A(u-\tau_n) X_u du + M_t \in R^N. \tag{2.7}$$

In the book [2] we show that if τ is a jump time with

$$F_t = P(\tau > t)$$

then

$$I_{t \geq \tau} + \int_0^{t \wedge \tau} \frac{dF_u}{F_{u-}}$$

is a martingale. The martingale property of $M_t \in R^N$ is a consequence of this result for each component.

3. Occupation Times

Consider a finite state, homogeneous semi-Markov chain $X = \{X_t,\, t \geq 0\}$ with the semimartingale representation as above:

$$X_t = X_{\tau_n} + \int_{\tau_n}^{\tau_{n+1} \wedge t} A(u-\tau_n) X_u du + M_t \in R^N. \tag{3.1}$$

For $1 \leq i \leq N$ define the process h_t^i, $t \geq 0$, by

$$h_t^i = \int_0^t \langle X_u, e_i \rangle \, du + \int_0^t h_{u-}^i \langle X_{u-}, e_i \rangle \langle e_i, dX_u \rangle \, . \tag{3.2}$$

Suppose initially the chain X is in state e_i . The second integral only contributes when X jumps from e_i to another state e_j with $i \neq j$. At the first jump time it contributes the amount

$$-h_{\tau_1 -}^i \langle X_{\tau_1 -}, e_i \rangle \, .$$

As $X_{\tau_1 -} = e_i$ this is $-h_{\tau_1 -}^i$.

However, the value of the first integral at $\tau_1 -$ is

$$h_{\tau_1 -}^i = \int_0^{\tau_1} \langle X_u, e_i \rangle \, du = \tau_1 \, .$$

Consequently,

$$\begin{aligned} h^i_{\tau_1} &= h^i_{\tau_1 -} - h^i_{\tau_1 -} \langle X_{\tau_1 -}, e_i \rangle \\ &= 0. \end{aligned}$$

A similar argument shows that at all subsequent jump times τ_k , $h^i_{\tau_k} = 0$.

Therefore, $h^i_{\tau_k}$ is re-set to 0 at each jump time τ_k so h^i_t measures the amount of time the chain has spent in state e_i since it last jumped to e_i . That is, $h^i_t = (t - \tau_k)$, given $X_{\tau_k} = e_i$ and $\tau_k \leq t < \tau_{k+1}$. Consider

$$h_t := \sum_{i=1}^N h^i_t . \tag{3.3}$$

Then

$$h_t = t + \int_0^t h_{u-} \langle X_{u-} , dX_u \rangle . \tag{3.4}$$

A similar argument shows that $h_t = t - \tau_k$ where $\tau_k \leq t < \tau_{k+1}$.

Consequently, we can write

$$\begin{aligned} a_{ji}(t - \tau_k) &= a_{ji}(h^i_t) = -p_{ji} \frac{g_i(h^i_t)}{G_i(h^i_t)} , \; i \neq j, \\ a_{ii}(t - \tau_k) &= a_{ii}(h^i_t) = \frac{g_i(h^i_t)}{G_i(h^i_t)} , \\ \text{and} \quad A(h_t) &= \big(a_{ji}(h^i_t)\big). \end{aligned}$$

The semimartingale dynamics of X are then

$$X_t = X_0 + \int_0^t A(h_u) X_u du + M_t \in R^N. \tag{3.5}$$

4. Semi-Markov Chain Generator

As above, $X = \{X_t, \, t \geq 0\}$ is a finite state continuous time homogeneous semi-Markov chain defined on the probability space $(\Omega, \mathcal{F}, P)$. The state space is the set $S = \{e_1, e_2, \ldots, e_N\}$ of unit vectors in R^N. h_t is the occupation time in the current state since the last jump. We have established the following dynamics for X and h :

$$X_t = X_0 + \int_0^t A(h_u) X_u du + M_t \in R^N \tag{4.1}$$

$$h_t = \; t + \int_0^t h_{u-} \langle X_{u-} , dX_u \rangle \in R. \tag{4.2}$$

From (4.1) and (4.2) we see the process (X_t, h_t) is Markov.

We shall use the following differentiation rules for semimartingales with no continuous martingale part. The proofs are in [2] Chapter 14. If $Z = \{Z_t, t \geq 0\}$ is a scalar semimartingale with no continuous martingale component and F a differentiable function

$$F(Z_t) = F(Z_0) + \int_0^t F'(Z_{u-}) dZ_u + \sum_{0<u\leq t} [F(Z_u) - F(Z_{u-}) - F'(Z_{u-}) \Delta Z_u]. \tag{4.3}$$

For two such semimartingales:

$$Y_tZ_t = Y_0Z_0 + \int_0^t Y_{u-}dZ_u + \int_0^t Z_{u-}dY_u + [Y,Z]_t \tag{4.4}$$

where

$$[Y,Z]_t = \sum_{0<u\leq t} \Delta Y_u \Delta Z_u\,.$$

Recall from Section (2.2) that

$$A(h_t) = (a_{ji}\,,\ 1 \leq i,\, j \leq N)$$

where for $i \neq j$

$$a_{ji}(t-\tau_k) = a_{ji}(h_t) = \frac{-p_{ji}g_i(h_t^i)}{G_i(h_t^i)}$$

and for $i = j$

$$a_{ii}(t-\tau_k) = a_{ii}(h_t) = \frac{g_i(h_t^i)}{G_i(h_t^i)}\,.$$

Recall the state space of X is the set

$$S = \{e_1, e_2, \dots, e_N\}$$

of unit vectors in R^N.

Suppose $\phi : S \times R \to R$ is an arbitrary C^1 function. Then for $\Delta \geq 0$:

$$\begin{aligned}\phi(X_{t+\Delta}, h_{t+\Delta}) - \phi(X_t, h_t) = &\int_t^{t+\Delta} \phi'(X_u, h_u)du \\ &+ \sum_{t<u\leq t+\Delta} \big(\phi(X_u, h_u) - \phi(X_{u-}, h_{u-})\big).\end{aligned} \tag{4.5}$$

Now

$$\big(\phi(X_u, h_u) - \phi(X_{u-}, h_{u-})\big) = \sum_{\substack{i,j=1\\ i\neq j}}^{N} \big(\phi(X_u, h_u) - \phi(X_{u-}, h_{u-})\big)(X_u, e_j)(X_{u-}, e_i).$$

For $i \neq j \quad \langle X_u, e_j\rangle \langle X_{u-}, e_i\rangle = \langle X_u - X_{u-}, e_j\rangle \langle X_{u-}, e_i\rangle$ so

$$\begin{aligned}&\big(\phi(X_u, h_u) - \phi(X_{u-}, h_{u-})\big) \langle X_u, e_j\rangle \langle X_{u-}, e_i\rangle \\ &\qquad = \big(\phi(e_j, 0) - \phi(e_i, h_{u-}\big) \langle \Delta X_u, e_j\rangle \langle X_{u-}, e_i\rangle \\ &\qquad = \big(\phi(e_j, 0)\big) - \phi(e_i, h_{u-})\big) \langle dX_u, e_j\rangle \langle X_{u-}, e_i\rangle.\end{aligned}$$

From (3.5)

$$dX_u = A(h_u)X_udu + dM_t$$

where M is a martingale so this is

$$= \big(\phi(e_j, 0) - \phi(e_i, h_{u-})\big) \langle A(h_u)X_udu + dM_u, e_j\rangle \langle X_{u-}, .e_i\rangle.$$

Therefore

$$\sum_{\substack{i,j=1\\i\neq j}}^{N} \big(\phi(X_u, h_{u-})\big) - \phi(X_{u-}, h_{u-})\big) \langle X_u, e_j\rangle \langle X_{u-}, e_i\rangle$$

$$= \sum_{\substack{i,j=1\\i\neq j}}^{N} \big(\phi(e_j, 0) - \phi(e_i, h_{u-})\big) \langle X_{u-}, e_i\rangle \langle A(h_u)X_u, e_j\rangle du$$

$$+ \text{ a martingale increment}$$

$$= \sum_{\substack{i,j=1\\i\neq j}}^{N} \big(\phi(e_j, 0) - \phi(e_i, h_{u-})\big) \langle X_u, e_i\rangle\, a_{ji}(h_u) du$$

$$+ \text{ a martingale increment.}$$

Consequently, assuming $X_t = e_i$,

$$\phi(X_{t+\Delta}, h_{t+\Delta}) - \phi(X_t, h_t)$$

$$= \int_t^{t+\Delta} \phi'(X_u, h_u)du + \sum_{\substack{i,j=1\\i\neq j}}^{N} \int_t^{t+\Delta} \big(\phi(e_j, 0) - \phi(e_i, h_{u-})\big) \langle X_u, e_i\rangle\, a_{ji}(h_u)du$$

$$+ \text{ a martingale term.}$$

From this we immediately deduce:

Theorem 4.1. *The generator of the Markov process* (X_t, h_t) *is*

$$\lim_{\Delta\downarrow 0} \frac{1}{\Delta} E\big[\phi(X_{t+\Delta}, h_{t+\Delta}) - \phi(X_t, h_t)|X_t, h_t\big]$$

$$= \phi'(X_t, h_t) + \sum_{\substack{j=1\\j\neq i}}^{N} \big(\phi(e_j, 0) - \phi(X_t, h_t)a_{ji}(h_t). \tag{4.6}$$

5. Density Dynamics

Notation 5.1. *Write*

$$q_s^i(y)dy = P\big(h_s \in [y, y+dy] \quad \text{and } X_s = e_i\,|X_0\big) \tag{5.1}$$

and

$$q_s(y) = \big(q_s^1(y), q_s^2(y), \dots, q_s^N(y)\big)' \in R^N. \tag{5.2}$$

Remark 5.2. Suppose $0 < \tau_1 < \tau_2 < \cdots < \tau_n < \dots$ are the jump times of X. Write $\{\tau_k^i\}$ for the jump times when $X_{\tau_k^i} = e_i$. Then

$$\int_0^t f'(h_{s-})\,\langle X_{s-}, e_i\,\rangle\, ds = \sum_{\tau_k^i \leq t} \int_{\tau_k^i}^{\tau_{k'}\wedge t} f'(s - \tau_k^i)ds$$

where $\tau_{k'}$ is the next jump time after τ_k^i

$$= \sum_{\tau_k^i \le t} \big(f(h_{\tau_{k'} \wedge t}) - f(h_{\tau_k^i}) \big).$$

If $\tau_k^i \le t < \tau_{k'}$ this term is

$$\big(f(t - \tau_k^i) - f(0) \big).$$

If $\tau_k^i < \tau_{k'} \le t$ this term is

$$\big(f(\tau_{k'} - \tau_k^i) - f(0) \big).$$

Also, if s is a jump time, τ_n , then $h_s = 0$ so

$$f(h_s) - f(h_{s-}) = f(0) - f(h_{s-}).$$

Theorem 5.3. *The density q^i satisfies the dynamics*

$$\begin{aligned} q_t^i(y) = \delta_0(y) - \int_0^\infty q_t^i(y) dy - \int_0^t \frac{\partial q_s^i(y)}{\partial y}\, ds \\ - \delta_0(y) \int_0^t \Big(\int_0^\infty a_{ii}(y) q_s^i(y) dy \Big) ds + \int_0^t a_{ii}(y) q_s^i(y) ds. \end{aligned} \tag{5.3}$$

Proof. Suppose f is an arbitrary differentiable function with compact support. Then using (4.3) and (4.4):

$$\begin{aligned} f(h_t) \langle X_t, e_i \rangle = f(0) \langle X_0, e_i \rangle + \int_0^t f(h_s) \langle A(h_s) X_s, e_i \rangle\, ds \\ + \int_0^t f(h_{s-}) \langle dM_s, e_i \rangle + \int_0^t f'(h_{s-}) \langle X_{s-}, e_i \rangle\, ds \\ + \sum_{0<s\le t} [f(h_s) - f(h_{s-})] \langle -X_{s-}, \Delta X_s \rangle \langle X_{s-}, e_i \rangle \\ + \sum_{0<s\le t} [f(h_s) - f(h_{s-})] \langle -X_{s-}, \Delta X_s \rangle \langle \Delta X_s, e_i \rangle\,. \end{aligned} \tag{5.4}$$

Now

$$\langle -X_{s-}, \Delta X_s \rangle \langle \Delta X_s, e_i \rangle = \langle \Delta X_s\,, e_i \rangle$$

so the last sum in (5.4) is:

$$\sum_{0<s\le t} [f(h_s) - f(h_{s-})] \langle \Delta X_s\,, e_i \rangle\,.$$

If there is a jump at s $h_s = 0$ so this is

$$\int_0^t [f(0) - f(h_{s-})] \langle dX_s\,, e_i \rangle$$

as X is piecewise constant. From (3.5) this is

$$f(0) \langle X_t - X_0\,, e_i \rangle - \int_0^t f(h_{s-}) \langle A(h_s) X_s ds + dM_s\,, e_i \rangle\,.$$

The integral here cancels the first two integrals in (5.4) so:

$$\begin{aligned} f(h_t)\langle X_t\,, e_i\,\rangle = f(0)\langle X_t\,, e_i\,\rangle &+ \int_0^t f'(h_{s-})\,\langle X_{s-}\,, e_i\,\rangle\, ds \\ &+ \int_0^t [f(0) - f(h_{s-})]\langle X_{s-}\,, e_i\,\rangle\,\langle -e_i\,, dX_s\,\rangle\,. \end{aligned}$$

The final integral here is

$$\begin{aligned} &-\int_0^t [f(0) - f(h_{s-})]\,\langle X_{s-}\,, e_i\,\rangle\,\langle A(h_s)X_s ds + dM_s\,, e_i\,\rangle \\ &\qquad = -f(0)\int_0^t a_{ii}(h_s)\,\langle X_s\,, e_i\,\rangle\, ds + \int_0^t f(h_s)a_{ii}(h_s)\,\langle X_s\,, e_i\,\rangle\, ds \\ &\qquad\quad + \text{ Martingale.} \end{aligned}$$

Consequently,

$$\begin{aligned} f(h_t)\,\langle X_t\,, e_i\,\rangle = f(0)\,\langle X_t\,, e_i\rangle &+ \int_0^t f'(h_s)\,\langle X_s\,, e_i\,\rangle ds \\ &- f(0)\int_0^t a_{ii}(h_s)\,\langle X_s\,, e_i\rangle\, ds + \int_0^t f(h_s)a_{ii}(h_s)\,\langle X_s\,, e_i\rangle\, ds \\ &+ \text{ Martingale.} \end{aligned} \tag{5.5}$$

Recall $q_s^i(y)$ is the probability density associated with $h_s\,\langle X_s\,, e_i\,\rangle$, given X_0, so for example

$$E[f(h_t)\,\langle X_t\,, e_i\,\rangle\,|X_0] = \int_0^\infty f(y)q_t^i(y)dy.$$

Note the domain of $q_t^i(y)$ is the range of h_t which is at most $[0,t]$; however, we write the y integrals over $[0,\infty)$. Therefore, taking expectations in (5.5), given X_0, we have

$$\begin{aligned} \int_0^\infty f(y)q_t^i(y)dy = f(0)\int_0^\infty q_t^i(y)dy &+ \int_0^t\int_0^\infty f'(y)q_s^i(y)dyds \\ &- f(0)\int_0^t \Big(\int_0^\infty a_{ii}(y)q_s^i(y)dy\Big)ds \\ &+ \int_0^t\int_0^\infty \big(f(y)a_{ii}(y)q_s^i(y)dy\big)ds. \end{aligned}$$

Integrating by parts, with $q^i_s(0) = q^i_s(\infty) = 0$ we have

$$\begin{aligned}\int_0^\infty f(y) q^i_t(y) dy = f(0) \int_0^\infty q^i_t(y) dy - \int_0^t \Big(\int_0^\infty f(y) \frac{\partial q^i_s(y)}{dy} dy \Big) ds \\ - f(0) \int_0^t \Big(\int_0^\infty a_{ii}(y) q^i_s(y) dy \Big) ds \\ + \int_0^t \Big(\int_0^\infty f(y) a_{ii}(y) q^i_s(y) dy \Big) ds.\end{aligned}$$

As f is arbitrary we see q^i satisfies

$$\begin{aligned}q^i_t(y) = \delta_0(y) \int_0^\infty q^i_t(y) dy - \int_0^t \frac{\partial q^i_s(y)}{\partial y} ds \\ - \delta_0(y) \int_0^t \Big(\int_0^\infty a_{ii}(y) q^i_s(y) dy \Big) ds + \int_0^t a_{ii}(y) q^i_s(y) ds.\end{aligned}$$

□

Corollary 5.4. *Taking marginals*

$$\begin{aligned}\int_0^\infty q_t(y) dy &= \Big(\int_0^\infty q^1_t(y) dy, \int_0^\infty q^2_t(y) dy, \dots, \int_0^\infty q^N_t(y) dy \Big)' \\ &= E[X_t | X_0]\end{aligned} \tag{5.6}$$

and

$$\sum_{i=1}^\infty q^i_t(y) dy = p(h^i_t \in [y, y + dy] \,|X_0). \tag{5.7}$$

6. Approximations

Suppose $X = \{X_t,\, t \geq 0\}$ is a finite state, continuous time homogeneous semi-Markov chain.

Equation (5.7) gives an expression for

$$E[X_t | X_0].$$

However, its calculation using the density $q_t(y)$ is not straightforward.

We now suggest an approximate procedure. With

$$p^i_t = P(X_t = e_i | X_0) = E[\langle X_t, e_i \rangle | X_0]$$

we have

$$p_t := (p^1_t, p^2_t, \dots, p^N_t)' = E[X_t | X_0] \in R^N.$$

Write $\widehat{p}_t$ and $\widehat{h}_t$ for $\mathcal{F}_t$-measurable approximations of p_t and h_t, respectively. Consider a discretization $0 < \delta < 2\delta < \cdots < n\delta < (n + 1)\delta < \dots$ of the time variable.

Suppose $t = n\delta$. Then from (3.5)

$$X_{t+\delta} = X_t + \int_t^{t+\delta} A(h_u) X_u du + (M_{t+\delta} - M_t).$$

Taking expectations, as $E[M_{t+\delta} - M_t|\mathcal{F}_t] = 0 \in R^N$

$$p_{t+\delta} = p_t + \int_t^{t+\delta} E[A(h_u)X_u|X_0]du. \tag{6.1}$$

Suppose we have $\mathcal{F}_t$-measurable approximations $\widehat{h}_t$ and $\widehat{p}_t$ of h_t and p_t have been found. Then from (5.1) an approximation for $p_{t+\delta}$ is

$$\widehat{p}_{t+\delta} := \widehat{p}_t + A(\widehat{h}_t)\widehat{p}_t\,\delta \in R^N. \tag{6.2}$$

Write e_{i^*} for a related MAP estimate of X_t , that is a state e_{i^*} such that $\widehat{p}\,_t^{i^*} \geq p_t^j$ for all j, $1 \leq j \leq N$.

The recurrence (6.2) gives an expression for

$$\widehat{p}_{t+\delta} = \left(\widehat{p}\,^1_{t+\delta}, \widehat{p}\,^2_{t+\delta}, \dots, \widehat{p}\,^N_{t+\delta}\right)'.$$

Again write e_{j^*} for the related MAP estimate of $X_{t+\delta}$, that is the state e_{j^*} such that $\widehat{p}\,^{j^*}_{t+\delta} \geq \widehat{p}\,^{\ell}_{t+\delta}$ for all ℓ, $1 \leq \ell \leq N$.

We then set $\widehat{h}_{t+\delta} = \widehat{h}_t + \delta$ if $i^* = j^*$ or

$$\widehat{h}_{t+\delta} = 0 \quad \text{if} \quad i^* \neq j^*.$$

We proceed in this way so that at each time $t + kh$, if estimates $\widehat{p}_{t+kh}$ and $\widehat{h}_{t+kh}$ are known we can obtain estimates $\widehat{p}_{t+(k+1)h}$ and $\widehat{h}_{t+(k+1)h}$. This can be used in calculations involving p_t and h_t .

7. Conclusion

This paper discusses a continuous time, finite state, homogeneous semi-Markov chain X. Its probabilistic and dynamic properties can be described by either a rate matrix, (or Q-matrix), or a matrix which describes occupation times in the various states. The rate matrix gives the rates of jumping from one state to another. The relation between these two descriptions is derived and the semimartingale decomposition of the semi-Markov chain derived.

An equation giving the dynamics of the occupation times is obtained. The dynamics for the conditional expectation of the occupation time and state is then derived. Finally some approximations for the dynamics are developed. Semi-Markov processes often fit observed data better than Markov processes. The results of this paper might find applications in fields such as signal processing and financial modelling.

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