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Resonance Varieties of Pure Braid-Like Groups

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Resonance Varieties of Pure Braid-Like Groups

by

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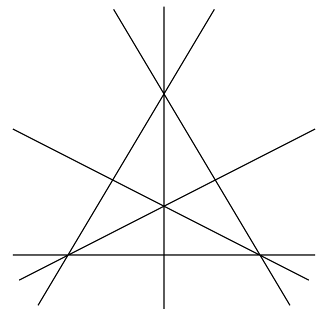


Figure 1: The braid arrangement $\mathcal{B}_4$

1 Introduction

In its simplest form, a hyperplane arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$ is a finite collection of lines in the real plane. The complement $M(\mathcal{A})$ of a real arrangement is a topological space composed of a finite number of disconnected regions. The problem of counting these regions dates back to the 19^{th} century. We can see that the number of regions created by an arrangement is a topological invariant, meaning that an arrangement can stretch or shrink without changing the number of regions. It turns out that the number of regions is the only interesting topological invariant of a real arrangement. Since every region is convex, each can be shrunk to a point, which is not topologically interesting. In 1975, Zaslavsky found a general counting formula, revealing that the number of regions depends only on the intersections of the arrangement. Thus, the topology of real arrangements and their complements is well understood.

Complex arrangements have a much richer topology than their real counterparts because removing a complex hyperplane does not disconnect complex space. This can be easily seen in the case $n = 1$: removing the origin, a 0-dimensional hyperplane, from $\mathbb{C}$ produces a space that can be shrunk to a circle. To describe the topological structure of the complement $M(\mathcal{A})$, we enlist cohomology. To a topological space X, we associate a sequence of abelian groups $H^k(X)$ called cohomology groups. These groups are a natural generalization of the region-counting of the real case; the rank of the 0^{th} cohomology group is the number of path-components of X. We will be interested in another topological invariant which is formed from the cohomology groups, namely the cohomology ring $H^*(X)$. In the case of complex arrangements, it turns out that $H^*(M(\mathcal{A}))$ depends only on the intersections of the arrangement.

This thesis is concerned with distinguishing topologically distinct spaces. To do so, we investigate $H^*(X)$ via the first resonance variety $\mathcal{R}^1(X)$. This topological invariant relates the cohomology groups that compose the cohomology ring within a cochain complex. This complex is an increasing sequence of cohomology groups connected by the cup product map

$$\cdots \longrightarrow H^{k-1}(X) \xrightarrow{\smile a} H^k(X) \xrightarrow{\smile a} H^{k+1}(X) \longrightarrow \cdots$$

We will see that $\mathcal{R}^1(X)$ is a stronger topological invariant than the cohomology groups. After calculating the first resonance variety of several topological spaces, we investigate a family $\pi_1(M_n)$ of "pure braid-like" groups by attempting to calculate their first resonance varieties.

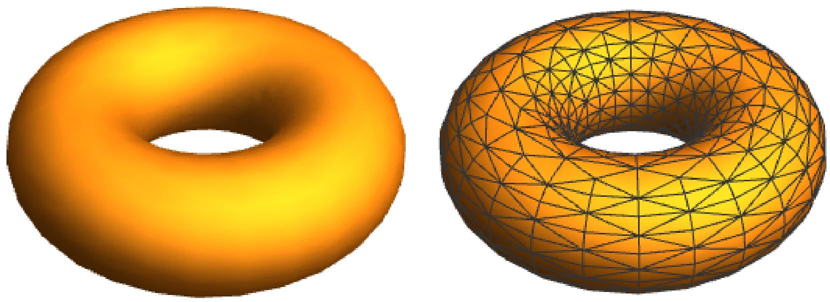


Figure 2: The torus (left) and a simplicial torus (right)

2 Basic Algebraic Topology

A fundamental problem of topology is to determine whether a given space can be deformed into another given space. In higher dimensions, where we cannot directly visualize spaces, we must rely on algebraic images to help us verify that two spaces are not the same. Algebraic topology can be thought of as the study of techniques for forming algebraic images of topological spaces. These algebraic images are most often groups, but they can also be more elaborate structures such as rings. In this section, we review some basic constructions of algebraic topology. For more details, see [Hat01].

2.1 Singular Homology Groups

A basic tool used to approximate topological spaces is the **n-simplex**, the n-dimensional analogue of a triangle (see Figure 2). A 0-simplex is a point, a 1-simplex is a closed interval, a 2-simplex is a triangle, and a 3-simplex is a tetrahedron. The **standard n-simplex** Δ^n is the n-simplex where every vertex agrees with the standard orthonormal basis $\{e_0, e_1, \ldots, e_n\}$ of $\mathbb{R}^{n+1}$:

$$\Delta^n = \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i \right\}.$$

A **singular n-simplex** in a topological space X is a continuous map $\sigma : \Delta^n \to X$. Let the n^{th} **chain group**

$$C_n(X) = \mathbb{Z}\{\sigma : \Delta^n \to X\}$$

be the free abelian group generated by the set of singular n-simplices in X. Elements of $C_n(X)$ can be written as finite sums $\sum_i n_i \sigma_i$ for $n_i \in \mathbb{Z}$ and $\sigma_i : \Delta^n \to X$. From the groups $C_n(X)$, we can form a **chain complex**, a sequence of homomorphisms $\partial_{n+1} : C_{n+1} \to C_n$ of abelian groups

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

such that $\partial_n \circ \partial_{n+1} \equiv 0$, i.e. $\operatorname{Im}\partial_{n+1} \subset \ker \partial_n$. Here, the boundary maps $\partial_n : C_n(X) \to C_{n-1}(X)$ are given by

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]},$$

where $\hat{v}_i$ indicates that v_i has been removed from the list, and $[v_0, \ldots, \hat{v}_i, \ldots, v_n]$ is the canonical identification with Δ^{n-1}, hence making $\sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]} : \Delta^{n-1} \to X$.

Definition 2.1. The n^{th} **homology group** of a topological space X is given by

$$H_n(X) = \ker \partial_n / \operatorname{Im}\partial_{n+1}.$$

Example 2.2. Let X be a point. We show that the only nonzero homology group of X is $H_0(X) \cong \mathbb{Z}$. For X, there is a unique n-simplex σ_n which collapses Δ^n to X, i.e. $\sigma_n(\Delta^n) = X$ for each $n \in \mathbb{N}$. Thus, each σ_n generates a chain group $C_n(X) \cong \mathbb{Z}$, and we have the chain complex

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{\partial_3} \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{\partial_0} 0$$

We see that $\partial_n(\sigma_n) = \sum_i (-1)^i \sigma_{n-1}$, which is an alternating sum of $(n+1)$ terms. Thus, for n odd, $\partial_n(\sigma_n) = 0$, and for n even, $\partial_n(\sigma_n) = \sigma_{n-1}$. In the even case, ∂_n is an isomorphism because it takes the generator of $C_n(X)$ to the generator of $C_{n-1}(X)$. Thus, we have the chain complex

$$\cdots \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} 0$$

This gives that $H_0(X) = \ker \partial_0 / \operatorname{Im}\partial_1 \cong \mathbb{Z}$ and that the rest of the homology groups are zero.

The topology of a point is not very interesting, as we alluded to in the introduction. Compare, for example, to the slightly more interesting homology of a circle:

$$H_n(S^1) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 1 \\ 0 & \text{else.} \end{cases}$$

The main purpose of homology is to distinguish between spaces of different homotopy types:

Proposition 2.3. *Let X, Y, be topological spaces. If there is a homotopy equivalence $f : X \to Y$, then $H_n(X) \cong H_n(Y)$ for all n.*

Thus, if two spaces have non-isomorphic homology groups, these spaces are not homotopy-equivalent. For example, a point is not homotopy-equivalent to a circle since

$$H_1(\text{pt.}) = 0 \neq \mathbb{Z} \cong H_1(S^1).$$

2.2 Cohomology Groups and Rings

Cohomology is dual to homology in the sense that n-chains C_n are replaced by their dual cochain group $C^n = \mathrm{Hom}(C_n, \mathbb{C})$. As before, these n-cochains form a **cochain complex**, a sequence of homomorphisms $\delta_n = \partial_n^* : C^n \to C^{n+1}$, where this sequence ascends rather than descends for duality reasons:

$$\cdots \leftarrow C^{n+1} \xleftarrow{\delta_n} C^n \xleftarrow{\delta_{n-1}} C^{n-1} \leftarrow \cdots \leftarrow C^1 \xleftarrow{\delta_0} C^0 \xleftarrow{\delta_{-1}} 0$$

These homomorphisms also satisfy $\delta_n \circ \delta_{n-1} \equiv 0$, which is equivalent to the inclusion $\mathrm{Im}\, \delta_{n-1} \subset \ker \delta_n$.

Definition 2.4. The n^{th} **cohomology group** of a topological space X is given by

$$H^n(X; \mathbb{C}) = \ker \delta_n / \mathrm{Im}\, \delta_{n-1}.$$

For convenience, we write $H^n(X) := H^n(X; \mathbb{C})$. As the dual of homology, cohomology is also a homotopy invariant:

Proposition 2.5. *Let X, Y, be topological spaces. If there is a homotopy equivalence $f : X \to Y$, then $H^n(X) \cong H^n(Y)$ for all n.*

What makes cohomology more powerful than homology is that the dualizing procedure used to construct the cohomology groups allows us to give these groups the structure of a graded ring. The product of this ring is the cup product, which is constructed in the following way.

Let $\varphi \in C^k(X), \psi \in C^l(X)$. The **cup product** $\varphi \smile \psi \in C^{k+l}(X)$ is the cochain whose value on a singular simplex $\sigma : \Delta^{k+l} \to X$ is given by the formula

$$(\varphi \smile \psi)(\sigma) = \varphi(\sigma|_{[v_0,\dots,v_k]})\psi(\sigma|_{[v_k,\dots,v_{k+l}]}).$$

A calculation shows that this map takes elements of $\ker \delta$ to $\ker \delta$ and takes elements of $\mathrm{Im}\, \delta$ to $\mathrm{Im}\, \delta$. Thus, it induces the cup product map in cohomology

$$\smile : H^k(X) \times H^l(X) \to H^{k+l}(X)$$

given by the formula

$$[\varphi] \smile [\psi] = [\varphi \cup \psi]$$

for each $\varphi \in C^k(X)$ and $\psi \in C^l(X)$.

Proposition 2.6. *Let $a \in H^k(X), b \in H^l(X)$. The cup product is* ***graded commutative****, meaning that*

$$a \smile b = (-1)^{kl} b \smile a \qquad \textit{and} \qquad a \smile b \in H^{k+l}(X)$$

As mentioned before, the cup product turns the cohomology groups into the graded ring $H^*(X) := (\bigoplus_{k\in\mathbb{N}} H^k(X), \smile)$. To indicate that an element $a \in H^*(X)$ lies in $H^k(X)$, we write $|a| = k$ for the **degree** of a. The identity element $1 \in H^0(X)$ for the cup product is the cohomology class of the cocycle $\varepsilon : C_0(X) \to \mathbb{C}$ that sends every singular 0-simplex to $1 \in \mathbb{C}$.

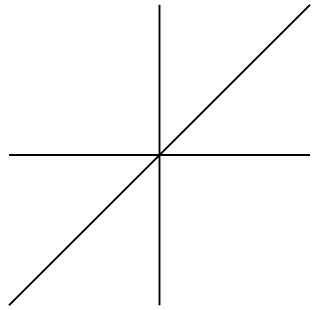


Figure 3: $P(\mathcal{A}) = xy(x-y)$

3 Hyperplanes

In this section, we review the basic combinatorial and algebraic tools in the theory of complex hyperplane arrangements. We are interested in the topology of arrangements as represented by the cohomology ring $H^*(M(\mathcal{A}))$, which is a measure of the topological complexity of $\mathcal{A}$. We will see that the structure of $H^*(M(\mathcal{A}))$ can be determined by an algebra that depends only on the intersections of the arrangement, namely the Orlik-Solomon algebra $A(\mathcal{A})$.

3.1 Basic Definitions

Definition 3.1. A complex **hyperplane** is an affine subspace of $\mathbb{C}^l$ with dimension $(l-1)$. An **arrangement** $\mathcal{A}$ is a finite set of hyperplanes in $\mathbb{C}^l$.

Each hyperplane $H \in \mathcal{A}$ is the kernel of a degree-one polynomial α_H defined up to a constant. Thus, we can specify an arrangement by the product

$$P(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H,$$

as in Figure 3. The most important topological invariant of $\mathcal{A}$ is its **complement**

$$M(\mathcal{A}) = \mathbb{C}^l \setminus \bigcup_{H \in \mathcal{A}} H.$$

As an open subset of $\mathbb{C}^l$, $M(\mathcal{A})$ is an open, smooth, connected manifold with complex dimension l. Also, $M(\mathcal{A})$ has the homotopy type of a finite CW-complex of complex dimension at most l. Thus, the cohomology groups of $M(\mathcal{A})$ are finite-dimensional and vanish in dimensions higher than l.

3.2 The Orlik-Solomon Algebra

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an arrangement in $\mathbb{C}^l$, and let $\mathbb{C}^n$ have basis $\{e_1, \ldots, e_n\}$ in correspondence with the hyperplanes. Then $\{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\}$

is a basis for the vector space $\bigwedge^k(\mathbb{C}^n)$. Let E be the **exterior algebra** over $\mathbb{C}^n$, i.e.

$$E = \mathbb{C} \oplus \mathbb{C}^n \oplus \bigwedge^2(\mathbb{C}^n) \oplus \bigwedge^3(\mathbb{C}^n) \oplus \cdots \oplus \bigwedge^n(\mathbb{C}^n).$$

Write $uv := u \wedge v$. The product $\wedge$ is graded-commutative, meaning that for $u \in \bigwedge^{|u|}(\mathbb{C}^n), v \in \bigwedge^{|v|}(\mathbb{C}^n)$, we have that

$$uv = (-1)^{|u||v|}vu, \qquad \text{and} \qquad uv \in \bigwedge^{|u|+|v|}(\mathbb{C}^n).$$

If $\{i_1, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$, we write $e_{i_1 \cdots i_k} = e_{i_1} \cdots e_{i_k}$. Define the derivation $\partial : E \to E$ by $\partial 1 = 0$, and for $k \geq 1$,

$$\partial e_{i_1 \cdots i_k} = \sum_{j=1}^{k} (-1)^{j-1} e_{i_1 \cdots \widehat{i_j} \cdots i_k},$$

where $\widehat{i_j}$ indicates removing e_{i_j} from the product.

For $S \subseteq \mathcal{A}$, denote the intersection of hyperplanes in S by $\cap S = \cap_{H \in S} H$. Also denote the codimension of $\cap S$ by $\operatorname{codim}(\cap S) = \dim \mathbb{C}^n - \dim \cap S$. We say that S is **dependent** if $\operatorname{codim}(\cap S) < |S|$. Geometrically, a collection of hyperplanes is dependent if the corresponding linear polynomials $\alpha_1, \cdots, \alpha_n$ are linearly dependent. For example, the arrangement in Figure 3 is dependent because $\operatorname{codim}(\cap \mathcal{A}) = 2 < |\mathcal{A}| = 3$. Alternatively, it is dependent because $(x - y)$ is a linear combination of x, y.

Definition 3.2. The **Orlik-Solomon algebra** of $\mathcal{A}$ is the quotient algebra

$$E/I(\mathcal{A}),$$

where $I(\mathcal{A}) \subset E$ is the ideal generated by

$$\{e_S \mid \cap S = \emptyset\} \cup \{\partial e_S \mid S \text{ is dependent}\}.$$

The OS-algebra offers the following topological interpretation in terms of the cohomology ring:

Theorem 3.3 (Orlik-Solomon)**.** *For an arrangement $\mathcal{A}$, there is a graded algebra isomorphism between the OS-algebra and the cohomology ring:*

$$A(\mathcal{A}) \xrightarrow{\cong} H^*(M(\mathcal{A})).$$

Thus, the structure of $H^*(M(\mathcal{A}))$ is determined by the structure of $A(\mathcal{A})$. In particular, the cohomology ring is generated only by elements of $H^1(M(\mathcal{A}))$, which are in correspondence with the hyperplanes of $\mathcal{A}$.

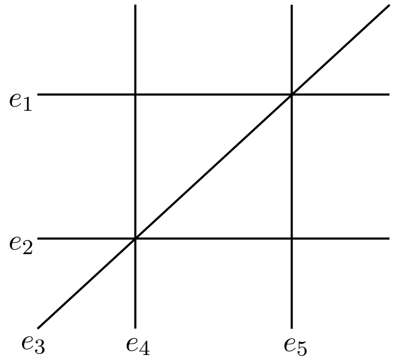


Figure 4

Example 3.4. Let $\mathcal{A}$ be the arrangement in $\mathbb{C}^2$ as shown above. The OS-algebra $A(\mathcal{A})$ is the quotient of the exterior algebra by the following relations:

$$e_{12} = e_{45} = 0, \quad \partial e_{234} = e_{34} - e_{24} + e_{23} = 0, \quad \partial e_{135} = e_{35} - e_{15} + e_{13} = 0.$$

From these relations, we can write

$$e_{34} = e_{23} - e_{23} \qquad e_{35} = e_{15} - e_{13}.$$

Thus, an ordered basis of $A(\mathcal{A})$ is given by

$$1 \qquad e_1, e_2, e_3, e_4, e_5 \qquad e_{13}, e_{14}, e_{15}, e_{23}, e_{24}, e_{25}$$

and

$$H^*(M(\mathcal{A})) \cong \mathbb{C} \oplus \mathbb{C}^5 \oplus \mathbb{C}^6.$$

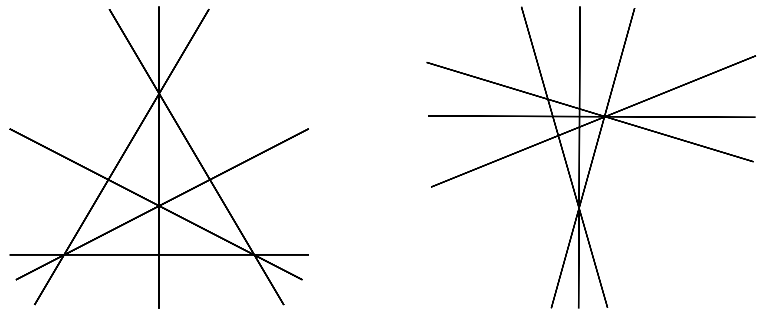


Figure 5: Arrangements with the same cohomology groups but nonisomorphic resonance varieties

4 Resonance Varieties

Resonance varieties are a topological invariant depending on the cohomology ring $H^*(X)$. We will see in Example 4.5 that the first resonance variety is a finer invariant than the cohomology groups. This means that two topological spaces can have the same cohomology groups but different resonance varieties, see Figure 5. This strength comes from the construction of the resonance variety as depending on maps between the graded pieces of $H^*(X)$.

4.1 Definitions and Properties

Noting that the complement $M(\mathcal{A})$ is path-connected as a connected manifold, we define the first resonance variety for a general path-connected topological space X. For $a \in H^1(X)$, the cup product $\smile a : H^n(X) \times H^1(X) \to H^{n+1}(X)$ is the boundary homomorphism for the cochain complex

$$0 \longrightarrow \mathbb{C} \xrightarrow{\smile a} H^1(X) \xrightarrow{\smile a} \cdots \xrightarrow{\smile a} H^n(X) \xrightarrow{\smile a} \cdots$$

Indeed, $a \smile a = -a \smile a = 0$ by graded-commutativity. We denote this complex by $(H^*(X), \smile a)$ and denote its cohomology by $H^*(H^*(X), \smile a)$.

Definition 4.1. The 1^{st} **resonance variety** of X is the set

$$\mathcal{R}^1(X) := \{a \in H^1(X) \mid H^1(H^*(X), \smile a) \neq 0\}.$$

Resonance varieties are also homotopy-type invariants:

Proposition 4.2. *Let X, Y, be topological spaces. If there is a homotopy equivalence $f : X \to Y$, then $\mathcal{R}^1(X) \cong \mathcal{R}^1(Y)$ as varieties.*

Proof. Let $f : X \to Y$ be a homotopy equivalence. Then the induced homomorphism in cohomology, $f^* : H^*(Y) \xrightarrow{\cong} H^*(X)$, defines isomorphisms between the

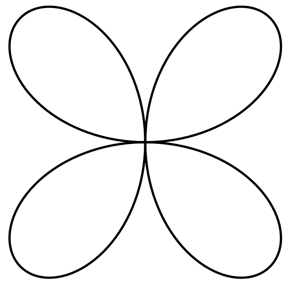


Figure 6: A wedge of four circles

cochain complexes for all $a \in H^1(Y)$:

$$(H^*(Y), \smile a) \xrightarrow{\cong} (H^*(X), \smile f^*(a)).$$

Thus, f^* restricts to an isomorphism $\mathcal{R}^1(Y) \xrightarrow{\cong} \mathcal{R}^1(X)$ of varieties. □

4.2 Examples

We begin with a simple topological calculation.

Example 4.3. Let $X = \bigvee_n S^1$ be a wedge of n circles, the topological space obtained by joining n copies of S^1 at a common basepoint. Note that X is homotopic to $\mathbb{C} \setminus \{n \text{ pts}\}$, which is an arrangement complement. The cohomology groups of X are given by

$$H^k(X) = \begin{cases} \mathbb{C} & \text{for } k = 0, \\ \mathbb{C}^n & \text{for } k = 1, \\ 0 & \text{else.} \end{cases}$$

So for $a \in \mathbb{C}^n$, we have the cochain complex

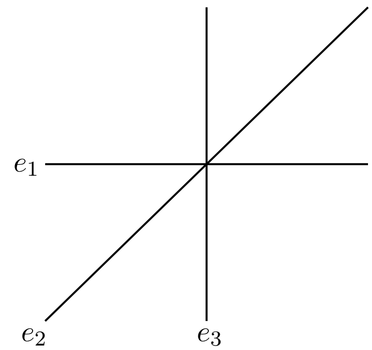
$$0 \longrightarrow \mathbb{C} \xrightarrow{\smile a} \mathbb{C}^n \xrightarrow{\smile a} 0$$

We see that $\ker\{\smile a : \mathbb{C}^n \to 0\} \cong \mathbb{C}^n$. Also, the image of the generator $1 \in \mathbb{C}$ is $1 \smile a = a \in \mathbb{C}^n$. So we have that

$$H^1(A(X), \smile a) = \mathbb{C}^n / \operatorname{Im}\{\smile a : \mathbb{C} \to \mathbb{C}^n\} = \begin{cases} \mathbb{C}^{n-1} & \text{for } a \neq 0, \\ \mathbb{C}^n & \text{for } a = 0. \end{cases}$$

Thus

$$\mathcal{R}^1(X) = \begin{cases} \mathbb{C}^n & \text{for } n > 1, \\ \{0\} & \text{for } n = 1. \end{cases}$$



Example 4.4. Let $\mathcal{A}$ be the arrangement in $\mathbb{C}^2$ as shown above. To calculate $\mathcal{R}^1(\mathcal{A})$, we must first calculate the OS-algebra $A(\mathcal{A})$. By definition, the OS-algebra $A(\mathcal{A})$ is the quotient of the exterior algebra by the following relation:

$$\partial e_{123} = e_{23} - e_{13} + e_{12} = 0.$$

From this relation, we can write

$$e_{23} = e_{13} - e_{12}.$$

Thus, an ordered basis of $A(\mathcal{A})$ is given by

$$1 \qquad e_1, e_2, e_3 \qquad e_{12}, e_{13} \tag{1}$$

For $a = a_1e_1 + a_2e_2 + a_3e_3 \in \mathbb{C}^3$, we consider the cochain complex

$$0 \longrightarrow \mathbb{C} \xrightarrow{\smile a} \mathbb{C}^3 \xrightarrow{\smile a} \mathbb{C}^2 \longrightarrow 0$$

For a nonzero element $a \in \mathbb{C}^3$ to be in $\mathcal{R}^1(\mathcal{A})$, it must satisfy

$$H^1(A, \smile a) := \ker\{\smile a : \mathbb{C}^3 \to \mathbb{C}^2\} / \operatorname{Im}\{\smile a : \mathbb{C} \to \mathbb{C}^3\} \neq 0.$$

Since 1 generates $\mathbb{C}$, and $1 \smile a = a$, we have that $\operatorname{Im}\{\smile a : \mathbb{C} \to \mathbb{C}^3\} \cong \mathbb{C}$. Thus, we need

$$\dim \ker\{\smile a : \mathbb{C}^3 \to \mathbb{C}^2\} \geq 2.$$

To examine the conditions on a which make the kernel large enough, we look at the image of the generators of $\mathbb{C}^3$ under the map $\smile a : \mathbb{C}^3 \to \mathbb{C}^2$.

$$\begin{aligned}
e_1 \mapsto e_1(a_1e_1 + a_2e_2 + a_3e_3) = a_1e_1^2 + a_2e_{12} + a_3e_{13} &= \mathbf{a_2}e_{12} + \mathbf{a_3}e_{13} \\
e_2 \mapsto e_2(a_1e_1 + a_2e_2 + a_3e_3) = a_1e_{21} + a_2e_2^2 + a_3e_{23} &= -a_1e_{12} + a_3(e_{13} - e_{12}) \\
&= -(\mathbf{a_1} + \mathbf{a_3})e_{12} + \mathbf{a_3}e_{13} \\
e_3 \mapsto e_3(a_1e_1 + a_2e_2 + a_3e_3) = a_1e_{31} + a_2e_{32} + a_3e_3^2 &= -a_1e_{13} - a_2(e_{13} - e_{12}) \\
&= \mathbf{a_2}e_{12} - (\mathbf{a1} + \mathbf{a2})e_{13}
\end{aligned}$$

Ordering bases as in (1), we can represent this map as a 2×3 matrix, where the bold coefficients correspond to a matrix entry:

$$\smile a = \begin{pmatrix} a_2 & -a_1 - a_3 & a_2 \\ a_3 & a_3 & -a_1 - a_2 \end{pmatrix}.$$

This matrix is equivalent (by column operations) to

$$\smile a = \begin{pmatrix} a_2 & a_1 + a_2 + a_3 & 0 \\ a_3 & 0 & a_1 + a_2 + a_3 \end{pmatrix}.$$

This matrix has three columns, so it has kernel with dimension at least two when at most one of the columns is independent. This implies that

$$a_1 + a_2 + a_3 = 0,$$

else the matrix would have two independent columns. If $a_2 = a_3 = 0$ as well, we have that $a_1 = a_2 = a_3 = 0$, which is a subspace of $a_1 + a_2 + a_3 = 0$. Thus,

$$\mathcal{R}^1(\mathcal{A}) = \{a \in \mathbb{C}^3 \mid a_1 + a_2 + a_3 = 0\}.$$

The following more general approach will be useful in future examples, where all calculations are done in the algebraic software Singular. The rank being at most one is equivalent to the condition that all 2×2 minors disappear. Each minor is a polynomial, and we can take the ideal generated by the three 2×2 minors, given by

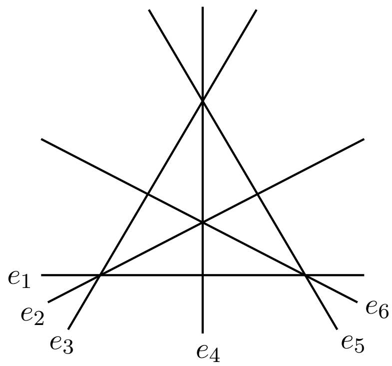
$$\langle -a_1^3 - a_1 a_2 - a_1 a_3,\ a_1 a_3 + a_2 a_3 + a_3^2,\ -a_1 a_2 - a_2^2 - a_2 a_3 \rangle.$$

Doing a primary decomposition of this ideal, we obtain

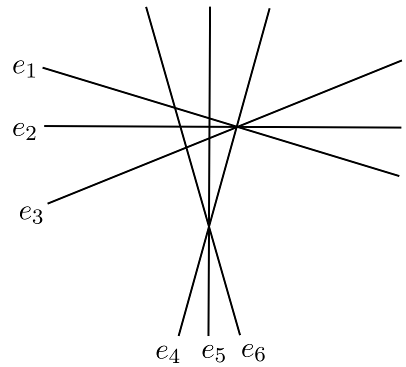
$$\langle a_1 + a_2 + a_3 \rangle \cap \langle a_1,\ a_2,\ a_3 \rangle.$$

Thus, the resonance variety is isomorphic to $V(a_1 + a_2 + a_3)$, so

$$\mathcal{R}^1(\mathcal{A}) = \{a \in \mathbb{C}^3 \mid a_1 + a_2 + a_3 = 0\}.$$



(a) $\mathcal{B}$



(b) $\mathcal{T}$

Example 4.5. Consider the braid arrangement $\mathcal{B}_4$ in $\mathbb{C}^4$ defined by

$$P(\mathcal{B}_4) = \prod_{1\le i<j\le 4} (x_i - x_j)$$

as well as the arrangement $\mathcal{T}_3$ in $\mathbb{C}^3$ defined by

$$P(\mathcal{T}_3) = x_1x_2x_3(x_1 + x_3)(x_1 + 2x_3)(x_2 + x_3).$$

Taking generic two-dimensional sections of $\mathcal{B}_4$ and $\mathcal{T}_3$, we respectively obtain $\mathcal{B}$ and $\mathcal{T}$ in the figure above. The following calculation will show that while $\mathcal{B}$ and $\mathcal{T}$ have the same cohomology groups, their resonance varieties differ.

The OS-algebra $A(\mathcal{B})$ is the quotient of the exterior algebra by the following relations:

$$\partial e_{123} = e_{23} - e_{13} + e_{12} = 0, \quad \partial e_{156} = e_{56} - e_{16} + e_{15} = 0,$$
$$\partial e_{246} = e_{46} - e_{26} + e_{24} = 0, \quad \partial e_{345} = e_{45} - e_{35} + e_{34} = 0,$$

$$e_{ijk} = 0 \text{ for } \{i,j,k\} \notin \{\{1,2,3\},\{1,5,6\},\{2,4,6\},\{3,4,5\}\}.$$

From these relations, we can write

$$e_{23} = e_{13} - e_{12}, \quad e_{45} = e_{35} - e_{34}, \quad e_{46} = e_{26} - e_{24}, \quad e_{56} = e_{16} - e_{15}.$$

Thus, a basis of $A(\mathcal{B})$ is given by

$$1 \qquad e_1, e_2, e_3, e_4, e_5, e_6 \qquad e_{12}, e_{13}, e_{14}, e_{15}, e_{16}, e_{24}, e_{25}, e_{26}, e_{34}, e_{35}, e_{36} \tag{2}$$

Similarly, the OS-algebra $A(\mathcal{T})$ is the quotient of the exterior algebra by the following relations:

$$\partial e_{123} = e_{23} - e_{13} + e_{12} = 0, \quad \partial e_{456} = e_{56} - e_{46} + e_{45} = 0,$$
$$\partial e_{145} = e_{45} - e_{15} + e_{14} = 0, \quad \partial e_{146} = e_{46} - e_{16} + e_{14} = 0,$$
$$\partial e_{156} = e_{56} - e_{16} + e_{15} = 0, \quad \partial e_{1456} = e_{456} - e_{156} + e_{146} - e_{145} = 0,$$

$$e_{ijk} = 0 \text{ for } \{i,j,k\} \notin \{\{1,2,3\},\{1,4,5\},\{1,4,6\},\{1,5,6\},\{4,5,6\}\}.$$

We note that the relation $\partial e_{1456} = 0$ is redundant since $\partial e_{1456} = (e_6 - e_1)(\partial e_{145})$, and $\partial e_{145} = 0$ already. From the other relations, we can write

$$e_{23} = e_{13} - e_{12}, \quad e_{45} = e_{15} - e_{14}, \quad e_{46} = e_{16} - e_{14}, \quad e_{56} = e_{16} - e_{15}.$$

Here, we omitted $\partial e_{456} = 0$ since $\partial e_{456} = e_{56} - e_{46} + e_{45}$, and each of these terms satisfies a relation above. Thus, a basis of $A(\mathcal{T})$ is given by

$$1 \qquad e_1, e_2, e_3, e_4, e_5, e_6 \qquad e_{12}, e_{13}, e_{14}, e_{15}, e_{16}, e_{24}, e_{25}, e_{26}, e_{34}, e_{35}, e_{36}$$

Since the graded pieces of the OS-algebra of both arrangements coincide, we see that $\mathcal{B}$ and $\mathcal{T}$ have isomorphic cohomology groups.

To distinguish $\mathcal{B}$ and $\mathcal{T}$, we calculate their resonance varieties. For $a = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6 \in \mathbb{C}^6$, we consider the cochain complex

$$0 \longrightarrow \mathbb{C} \xrightarrow{\smile a} \mathbb{C}^6 \xrightarrow{\smile a} \mathbb{C}^{11} \longrightarrow \cdots$$

For a nonzero element $a \in \mathbb{C}^6$ to be in $\mathcal{R}^1$, it must satisfy

$$H^1(A, \smile a) := \ker\{\smile a : \mathbb{C}^6 \to \mathbb{C}^{11}\} / \operatorname{Im}\{\smile a : \mathbb{C} \to \mathbb{C}^6\} \neq 0.$$

Since 1 generates $\mathbb{C}$, and $1 \smile a = a$, we have that $\operatorname{Im}\{\smile a : \mathbb{C} \to \mathbb{C}^6\} \cong \mathbb{C}$. Thus, we need

$$\dim \ker\{\smile a : \mathbb{C}^6 \to \mathbb{C}^{11}\} \geq 2.$$

To examine the conditions on a which make the kernel large enough, we look at the image of the generators of $\mathbb{C}^6$ under the map $\smile a : \mathbb{C}^6 \to \mathbb{C}^{11}$. For instance, for $\mathcal{B}$, we have that

$$\begin{aligned} e_2 \mapsto &- a_1e_{12} + a_3e_{23} + a_4e_{24} + a_5e_{25} + a_6e_{26} \\ =& - a_1e_{12} + a_3(e_{13} - e_{12}) + a_4e_{24} + a_5e_{25} + a_6e_{26} \\ =& -(\mathbf{a_1} + \mathbf{a_3})e_{12} + \mathbf{a_3}e_{13} + \mathbf{a_4}e_{24} + \mathbf{a_5}e_{25} + \mathbf{a_6}e_{26}. \end{aligned}$$

Ordering bases as in (2), we can represent these maps as matrices:

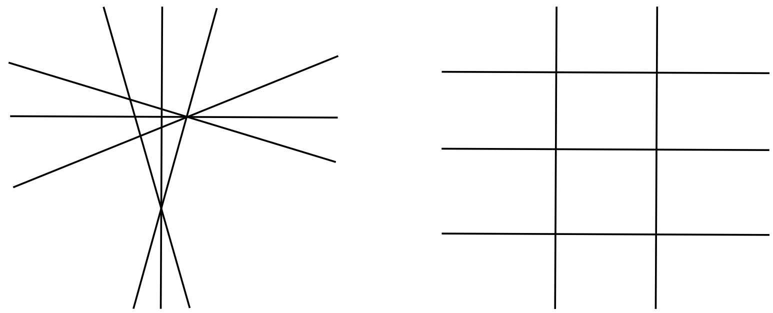
$$\smile_{\mathcal{B}} a = \begin{pmatrix} a_2 & -a_1-a_3 & a_2 & 0 & 0 & 0 \\ a_3 & a_3 & -a_1-a_2 & 0 & 0 & 0 \\ a_4 & 0 & 0 & -a_1 & 0 & 0 \\ a_5 & 0 & 0 & 0 & -a_1-a_6 & a_5 \\ a_6 & 0 & 0 & 0 & a_6 & -a_1-a_5 \\ 0 & a_4 & 0 & -a_2-a_6 & 0 & a_4 \\ 0 & a_5 & 0 & 0 & -a_2 & 0 \\ 0 & a_6 & 0 & a_6 & 0 & -a_2-a_4 \\ 0 & 0 & a_4 & -a_3-a_5 & a_4 & 0 \\ 0 & 0 & a_5 & a_5 & -a_3-a_4 & 0 \\ 0 & 0 & a_6 & 0 & 0 & -a_3 \end{pmatrix}$$

Similarly, we have the following matrix for $\smile a_{\mathcal{T}}$:

$$\begin{pmatrix} a_2 & -a_1-a_3 & a_2 & 0 & 0 & 0 \\ a_3 & a_3 & -a_1-a_2 & 0 & 0 & 0 \\ a_4 & 0 & 0 & -a_1-a_5-a_6 & a_4 & a_4 \\ a_5 & 0 & 0 & a_5 & -a_1-a_4-a_6 & a_5 \\ a_6 & 0 & 0 & a_6 & a_6 & -a_1-a_4-a_5 \\ 0 & a_4 & 0 & -a_2 & 0 & 0 \\ 0 & a_5 & 0 & 0 & -a_2 & 0 \\ 0 & a_6 & 0 & 0 & 0 & -a_2 \\ 0 & 0 & a_4 & -a_3 & 0 & 0 \\ 0 & 0 & a_5 & 0 & -a_3 & 0 \\ 0 & 0 & a_6 & 0 & 0 & -a_3 \end{pmatrix}$$

By the Rank-Nullity theorem, the sum of the dimension of the image and the dimension of the kernel equals the number of columns of the matrix. Thus, this matrix has kernel with dimension at least 2 when it has rank at most 4. This rank-condition is equivalent to the condition that all 5×5 minors disappear. Performing calculations as in Example 4.4, we obtain that

$$\begin{aligned} R^1(\mathcal{B}) = &\{a \in \mathbb{C}^6 \mid a_1 + a_5 + a_6 = a_2 = a_3 = a_4 = 0\} \\ \cup\ &\{a \in \mathbb{C}^6 \mid a_2 + a_4 + a_6 = a_1 = a_3 = a_5 = 0\} \\ \cup\ &\{a \in \mathbb{C}^6 \mid a_3 + a_4 + a_5 = a_1 = a_2 = a_6 = 0\} \\ \cup\ &\{a \in \mathbb{C}^6 \mid a_1 + a_2 + a_3 = a_4 = a_5 = a_6 = 0\} \\ \cup\ &\{a \in \mathbb{C}^6 \mid a_4 + a_5 + a_6 = a_1 - a_4 = a_2 - a_5 = a_3 - a_6 = 0\}, \end{aligned}$$



(a) $\mathcal{T}_3$ (b) $\mathbf{d}\boldsymbol{\mathcal{T}_3}$

Figure 8

while

$$\begin{aligned} R^1(\mathcal{T}) &= \{a \in \mathbb{C}^6 \mid a_1 + a_2 + a_3 = a_4 = a_5 = a_6 = 0\} \\ &\cup \ \{a \in \mathbb{C}^6 \mid a_1 + a_4 + a_5 + a_6 = a_2 = a_3 = 0\}. \end{aligned}$$

We see that $R^1(\mathcal{B})$ has five irreducible components. The first four components correspond to the subarrangements consisting of three intersecting hyperplanes, while the last compononent does not correspond to any subarrangement (a so-called *essential* component). However, $R^1(\mathcal{T})$ has only two irreducible components. The first component corresponds to the subarrangement consisting of three intersecting hyperplanes, while the second component corresponds to the subarrangement consisting of four intersecting hyperplanes.

Since their resonance varieties are different, $H^*(M(\mathcal{B})) \ncong H^*(M(\mathcal{T}))$ as rings, so the arrangements $\mathcal{B}$ and $\mathcal{T}$ are not homotopy equivalent.

Remark 4.6. The complement of the braid arrangement $\mathcal{B}_4$ is aspherical, and its fundamental group is the pure braid group on 4 strands P_4. It can be shown that P_4 is an iterated semidirect product of free groups, i.e.

$$P_4 \cong F_3 \rtimes F_2 \rtimes F_1.$$

On the other hand, the complement of the arrangement $\mathcal{T}_3$ is aspherical, and its fundamental group is $F_3 \times F_2 \times F_1$. To see this, we first *decone* $\mathcal{T}_3$, obtaining $\mathbf{d}\boldsymbol{\mathcal{T}_3}$, by choosing a hyperplane to be a hyperplane at infinity, as in Figure 8. The relationship between $\mathcal{T}_3$ and $\mathbf{d}\boldsymbol{\mathcal{T}_3}$ is given by

$$M(\mathcal{T}_3) \cong M(\mathbf{d}\boldsymbol{\mathcal{T}_3}) \times \mathbb{C}^*.$$

We see that $M(\mathcal{T}_3)$ is homeomorphic to $C_3 \times C_2 \times C_1$, where C_k denotes the complex lines with k points removed. Thus, $\pi_1(M(\mathcal{T}_3)) \cong F_3 \times F_2 \times F_1$.

In the previous example, $\mathcal{B}$ and $\mathcal{T}$ were generic two-dimensional sections of $\mathcal{B}_4$ and $\mathcal{T}_3$ respectively, so that $\mathcal{R}^1(\mathcal{B}) = \mathcal{R}^1(\mathcal{B}_4)$ and $\mathcal{R}^1(\mathcal{T}) = \mathcal{R}^1(\mathcal{T}_3)$. We saw that

$\mathcal{R}^1(\mathcal{B})$ and $\mathcal{R}^1(\mathcal{T})$ differ, so that $\mathcal{R}^1(\mathcal{B}_4)$ and $\mathcal{R}^1(\mathcal{T}_3)$ differ. Since $M(\mathcal{B}_4)$ and $M(\mathcal{T}_3)$ are aspherical, we have that

$$\mathcal{R}^1(\pi_1(M(\mathcal{B}_4))) = \mathcal{R}^1(\mathcal{B}_4) \neq \mathcal{R}^1(\mathcal{T}_3) = \mathcal{R}^1(\pi_1(M(\mathcal{T}_3))).$$

Thus,

$$P_4 \cong \pi_1(M(\mathcal{B}_4)) \ncong \pi_1(M(\mathcal{T}_3)) \cong F_3 \times F_2 \times F_1,$$

so that P_4 is not an iterated direct product of free groups. A more involved resonance variety calculation can show the same for all P_n for $n \geq 4$. This is in contrast to $P_2 \cong F_1$ and $P_3 \cong F_2 \times F_1$.

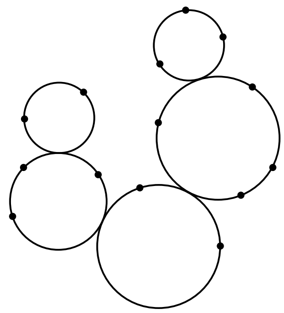


Figure 9: A stable curve with 14 marked points

5 The Pure Cactus Group

In this section, we study the resonance variety of a sequence of spaces $\{M_n\}$. The fundamental group $\pi_1(M_n)$ of M_n is called the pure cactus group, and it resembles the pure braid group PB_{n-1} in several ways. The definition of M_n is given below, but what is relevant to us is that the structure of the cohomology ring has recently been determined [EHKR]. Because M_n is aspherical, the cohomology of M_n coincides with the cohomology of $\pi_1(M_n)$, so we use them interchangeably.

Definition 5.1. A **stable rational curve** with n marked points is a finite union C of projective lines $C_1, \ldots, C_k$ together with distinct marked points $z_1, \ldots z_n$ such that

1. Each marked point z_i is in only one C_j.
2. $C_i \cap C_j$ is either empty or consists of one point, and in the latter case the intersection is transversal.
3. The graph of components (whose vertices are the lines C_i and whose edges correspond to pairs of intersecting lines) is a tree.
4. Each component C_i has at least three special points, where a special point is either an intersection with another component, or a marked point.

Over the real numbers, projective lines are circles, so a stable curve is a tree of circles with labeled points on them, see Figure 9.

For $n \geq 3$, let $M_n := \overline{\mathcal{M}}_{0,n}(\mathbb{R})$ be the real locus of the Deligne-Mumford compactification $\overline{\mathcal{M}}_{0,n}$ of the moduli space of rational curves with n marked points. It can be described as the set of equivalence classes of stable curves with n marked points defined over $\mathbb{R}$. It is known that M_3 is a point, M_4 is a circle, and M_5 is the connected sum of five copies of $\mathbb{RP}^1$.

Proposition 5.2. *The moduli space M_n is a connected, compact, smooth manifold of dimension $(n-3)$.*

As previously mentioned, $\pi_1(M_n)$ is a "pure braid-like" group, and the first resonance variety of the pure braid group is known. Thus, we might conjecture a similiar result for $\mathcal{R}^1(M_n)$.

Proposition 5.3 ([CS99]). *Let $l = \binom{n}{2}$. The first resonance variety of the pure braid group PB_n is given by*

$$\mathcal{R}^1(PB_n) = \bigcup_{1\leq i<j<k\leq n} V_{i,j,k} \cup \bigcup_{\substack{I\subset[n] \\ |I|=4}} V_I,$$

where

$$V_{i,j,k} = \{t \in \mathbb{C}^l \mid t_{i,j} + t_{i,k} + t_{j,k} = 0,\ \text{and } t_{p,q} = 0 \text{ if } |\{p,q\} \cap \{i,j,k\}| \leq 1\}$$

and

$$V_I = \{t \in \mathbb{C}^l \mid t_{i,j} - t_{p,q} = 0 \text{ if } \{i,j\} \cup \{p,q\} = I,\ t_{p,q} = 0 \text{ if } \{p,q\} \not\subset I,\ \text{and } T = 0\}$$

for $T = \sum_{1\leq i<j\leq n} t_{i,j}$.

5.1 Investigation into the resonance of M_n

To calculate the resonance variety $\mathcal{R}^1(M_n)$, we need a presentation of the cohomology ring of M_n, which is given below.

Theorem 5.4 ([EHKR]). *Let Λ_n be the skew-commutative algebra generated by the elements of degree one $\omega_{ijkl}, 1 \leq i,j,k,l \leq n$, with defining relations*

$$\omega_{ijkl}\omega_{ijkm} = 0$$

and

$$\omega_{ijkl} + \omega_{jklm} + \omega_{klmi} + \omega_{lmij} + \omega_{mijk} = 0,$$

where the ω_{ijkl} are antisymmetric in i,j,k,l[1]*. Then*

$$H^*(M_n) \cong \Lambda_n.$$

Using the linear linear relation from Theorem 5.4 with $m = n$, the generators ω_{ijkl} can be written as

$$\omega_{ijkl} = \omega_{ijkn} - \omega_{jkln} + \omega_{klin} - \omega_{lijn},$$

so that

$$\{\omega_{ijkn} \mid 1 \leq i < j < k \leq n-1\}$$

is a generating set for Λ_n. We will be working with the following presentation which has only quadratic relations.

[1] This means that $\omega_{ijkl} = -\omega_{jikl} = -\omega_{ikjl} = -\omega_{ijlk}$.

Proposition 5.5. *Let Λ_n be the skew-commutative algebra generated by the elements of degree one $\nu_{ijk}, 1 \leq i, j, k \leq n-1$, with defining relations*

$$\nu_{ijk}\nu_{ijl} = 0$$

and

$$\nu_{ijk}\nu_{klm} + \nu_{jkl}\nu_{lmi} + \nu_{klm}\nu_{mij} + \nu_{lmi}\nu_{ijk} + \nu_{mij}\nu_{jkl} = 0,$$

where the ν_{ijk} are antisymmetric in i, j, k. Then

$$H^*(M_n) \cong \Lambda_n.$$

All elements of $H^*(M_n)$ are obtained from products of generators ν_{ijk} of $H^1(M_n)$ for $1 \leq i, j, k \leq n-1$. To calculate the first resonance variety, we first form a basis for $H^1(M_n)$ and $H^2(M_n)$. From this basis, we form the cochain complex

$$0 \longrightarrow \mathbb{C} \xrightarrow{\smile a} H^1(M_n) \xrightarrow{\smile a} H^2(M_n) \longrightarrow \cdots$$

In the case $n = 3$, we see that $H^1(M_3) = 0$ since there are no nontrivial ν_{ijk} for $1 \leq i, j, k \leq 2$ by antisymmetry. So $\mathcal{R}^1(M_3) = \emptyset$.

In the case $n = 4$, $M_4 \cong S^1$. Indeed, we have the single generator ν_{123} with no nontrivial relations. For $a \in \mathbb{C}$, we have the cochain complex

$$0 \longrightarrow \mathbb{C} \xrightarrow{\smile a} \mathbb{C} \xrightarrow{\smile a} 0$$

The first cohomology of this complex is given by

$$H^1(H^*(M_4), \smile a) = \mathbb{C}/\operatorname{Im}\{\smile a : \mathbb{C} \to \mathbb{C}\}.$$

Thus, $H^1(H^*(M_4), \smile a) \neq 0$ only when $a = 0$, so $\mathcal{R}^1(M_4) = \{0\}$.

In the case $n = 5$, M_5 is the connected sum of five real projective planes. We have the generators $\nu_{123}, \nu_{124}, \nu_{134}, \nu_{234}$, with the relations

$$\nu_{123}\nu_{124} = \nu_{123}\nu_{134} = \nu_{123}\nu_{234} = \nu_{124}\nu_{134} = \nu_{124}\nu_{234} = \nu_{134}\nu_{234} = 0.$$

Since these relations are all possible combinations of two generators, the second cohomology is zero. Thus, we have the cochain complex

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}^4 \longrightarrow 0$$

The first cohomology of this complex is given by

$$H^1(H^*(M_5), \smile a) = \mathbb{C}^4/\operatorname{Im}\{\smile a : \mathbb{C} \to \mathbb{C}^4\}.$$

Thus, $H^1(H^*(M_4), \smile a) \neq 0$ for all $a \in \mathbb{C}^4$, so $\mathcal{R}^1(M_5) = \mathbb{C}^4$.

Collecting what we have seen so far, we have that

$$\mathcal{R}^1(M_n) = \begin{cases} \emptyset & \text{for } n = 3, \\ \{0\} & \text{for } n = 4, \\ \mathbb{C}^4 & \text{for } n = 5. \end{cases}$$

Moving on to M_6, we have the generators

$$\nu_{123}, \nu_{124}, \nu_{125}, \nu_{134}, \nu_{135}, \nu_{145}, \nu_{234}, \nu_{235}, \nu_{245}, \nu_{345}$$

with the thirty monomial relations

$$\begin{aligned} \nu_{123}\nu_{124} = \nu_{123}\nu_{125} = \nu_{123}\nu_{134} = \nu_{123}\nu_{135} = \nu_{123}\nu_{234} = \nu_{123}\nu_{235} = \nu_{124}\nu_{125} = {} & 0, \\ \nu_{124}\nu_{134} = \nu_{124}\nu_{145} = \nu_{124}\nu_{234} = \nu_{124}\nu_{245} = \nu_{125}\nu_{135} = \nu_{125}\nu_{145} = \nu_{125}\nu_{235} = {} & 0, \\ \nu_{125}\nu_{245} = \nu_{134}\nu_{135} = \nu_{134}\nu_{145} = \nu_{134}\nu_{234} = \nu_{134}\nu_{345} = \nu_{135}\nu_{145} = \nu_{135}\nu_{235} = {} & 0, \\ \nu_{135}\nu_{345} = \nu_{145}\nu_{245} = \nu_{145}\nu_{345} = \nu_{234}\nu_{235} = \nu_{234}\nu_{245} = \nu_{234}\nu_{345} = \nu_{235}\nu_{245} = {} & 0, \\ \nu_{235}\nu_{345} = \nu_{245}\nu_{345} = {} & 0, \end{aligned}$$

and with the six nonlinear relations

$$\begin{aligned} \nu_{123}\nu_{345} &= -\nu_{234}\nu_{451} - \nu_{345}\nu_{512} - \nu_{451}\nu_{123} - \nu_{512}\nu_{234} \\ &= \nu_{123}\nu_{145} - \nu_{125}\nu_{234} + \nu_{125}\nu_{345} + \nu_{145}\nu_{234}, \end{aligned}$$

$$\begin{aligned} \nu_{135}\nu_{245} = \nu_{135}\nu_{524} &= -\nu_{352}\nu_{241} - \nu_{524}\nu_{413} - \nu_{241}\nu_{135} - \nu_{413}\nu_{352} \\ &= -\nu_{124}\nu_{135} + \nu_{124}\nu_{235} - \nu_{134}\nu_{235} + \nu_{134}\nu_{245}, \end{aligned}$$

$$\begin{aligned} \nu_{145}\nu_{235} = \nu_{145}\nu_{523} &= -\nu_{452}\nu_{231} - \nu_{523}\nu_{314} - \nu_{231}\nu_{145} - \nu_{314}\nu_{452} \\ &= -\nu_{123}\nu_{145} + \nu_{123}\nu_{245} - \nu_{134}\nu_{235} + \nu_{134}\nu_{245}, \end{aligned}$$

$$\begin{aligned} \nu_{124}\nu_{345} = \nu_{124}\nu_{453} &= -\nu_{245}\nu_{531} - \nu_{453}\nu_{312} - \nu_{531}\nu_{124} - \nu_{312}\nu_{245} \\ &= \nu_{123}\nu_{145} - \nu_{123}\nu_{245} - \nu_{124}\nu_{235} - \nu_{125}\nu_{234} \\ &\quad + \nu_{125}\nu_{345} + \nu_{134}\nu_{235} - \nu_{134}\nu_{245} + \nu_{145}\nu_{234}, \end{aligned}$$

$$\begin{aligned} \nu_{135}\nu_{234} = \nu_{153}\nu_{324} &= -\nu_{532}\nu_{241} - \nu_{324}\nu_{415} - \nu_{241}\nu_{153} - \nu_{415}\nu_{532} \\ &= \nu_{123}\nu_{145} - \nu_{123}\nu_{245} + \nu_{124}\nu_{135} - \nu_{124}\nu_{235} \\ &\qquad + \nu_{134}\nu_{235} - \nu_{134}\nu_{245} + \nu_{145}\nu_{234} \end{aligned}$$

$$\begin{aligned} \nu_{125}\nu_{134} = \nu_{251}\nu_{134} &= -\nu_{513}\nu_{342} - \nu_{134}\nu_{425} - \nu_{342}\nu_{251} - \nu_{425}\nu_{513} \\ &= -\nu_{123}\nu_{145} + \nu_{123}\nu_{245} + \nu_{125}\nu_{234} + \nu_{134}\nu_{245} \\ &\qquad - \nu_{145}\nu_{234} \end{aligned}$$

Thus, we have the ordered basis

$$1 \qquad \nu_{123}, \nu_{124}, \nu_{125}, \nu_{134}, \nu_{135}, \nu_{145}, \nu_{234}, \nu_{235}, \nu_{245}, \nu_{345} \tag{3}$$

$$\nu_{123}\nu_{145}, \nu_{123}\nu_{245}, \nu_{124}\nu_{135}, \nu_{124}\nu_{235}, \nu_{125}\nu_{234}, \nu_{125}\nu_{345}, \nu_{134}\nu_{235}, \nu_{134}\nu_{245}, \nu_{145}\nu_{234}.$$

This gives the cochain complex

$$0 \longrightarrow \mathbb{C} \xrightarrow{\smile a} \mathbb{C}^{10} \xrightarrow{\smile a} \mathbb{C}^{9} \longrightarrow 0$$

For a nonzero element $a \in \mathbb{C}^{10}$, written as

$$a_1\nu_{123} + a_2\nu_{124} + a_3\nu_{125} + a_4\nu_{134} + a_5\nu_{135} + a_6\nu_{145} + a_7\nu_{234} + a_8\nu_{235} + a_9\nu_{245} + a_{10}\nu_{345},$$

to be in $\mathcal{R}^1$, it must satisfy

$$H^1(A, \smile a) := \ker\{\smile a : \mathbb{C}^{10} \to \mathbb{C}^9\} / \operatorname{Im}\{\smile a : \mathbb{C} \to \mathbb{C}^{10}\} \neq 0.$$

As before, we need

$$\dim \ker\{\smile a : \mathbb{C}^{10} \to \mathbb{C}^9\} \geq 2.$$

To examine the conditions on a which make the kernel large enough, we look at the image of the generators of $\mathbb{C}^{10}$ under the map $\smile a : \mathbb{C}^{10} \to \mathbb{C}^9$. For instance, the generator

$$\begin{aligned}
\nu_{124} \mapsto\ & a_5\nu_{124}\nu_{135} + a_8\nu_{124}\nu_{235} + a_{10}\nu_{124}\nu_{345} \\
&= a_5\nu_{124}\nu_{135} + a_8\nu_{124}\nu_{235} + a_{10}(\nu_{123}\nu_{145} - \nu_{123}\nu_{245} - \nu_{124}\nu_{235} - \nu_{125}\nu_{234} \\
&\quad + \nu_{125}\nu_{345} + \nu_{134}\nu_{235} - \nu_{134}\nu_{245} + \nu_{145}\nu_{234}) \\
&= \mathbf{a_{10}}\nu_{123}\nu_{145} - \mathbf{a_{10}}\nu_{123}\nu_{245} + \mathbf{a_5}\nu_{124}\nu_{135} + (\mathbf{a_8} - \mathbf{a_{10}})\nu_{124}\nu_{235} - \mathbf{a_{10}}\nu_{125}\nu_{234} \\
&\quad + \mathbf{a_{10}}\nu_{125}\nu_{345} + \mathbf{a_{10}}\nu_{134}\nu_{235} - \mathbf{a_{10}}\nu_{134}\nu_{245} + \mathbf{a_{10}}\nu_{145}\nu_{234}
\end{aligned}$$

Ordering bases as in (3), we can represent this map by the 9×10 matrix A shown on Page 25.

$$\begin{pmatrix}
a_{10}+a_6 & a_{10} & -a_4 & a_3 & a_7 & -a_1-a_8 & -a_5 & a_6 & 0 & -a_1-a_2 \\
a_9 & -a_{10} & a_4 & -a_3 & -a_7 & a_8 & a_5 & -a_6 & -a_1 & a_2 \\
0 & a_5 & 0 & 0 & -a_2+a_7-a_9 & 0 & -a_5 & 0 & a_5 & 0 \\
0 & a_8-a_{10} & 0 & 0 & a_9-a_7 & 0 & a_5 & -a_2 & -a_5 & a_2 \\
-a_{10} & -a_{10} & a_4+a_7 & -a_3 & 0 & 0 & -a_3 & 0 & 0 & a_1+a_2 \\
a_{10} & a_{10} & a_{10} & 0 & 0 & 0 & 0 & 0 & 0 & -a_1-a_2-a_3 \\
0 & a_{10} & 0 & a_8 & a_7-a_9 & -a_8 & -a_5 & a_6-a_4 & a_5 & -a_2 \\
0 & -a_{10} & a_4 & a_9-a_3 & a_9-a_7 & a_8 & a_5 & -a_6 & -a_4-a_5 & a_2 \\
a_{10} & a_{10} & -a_4 & a_3 & a_7 & a_7 & -a_5-a_6 & 0 & 0 & -a_1-a_2
\end{pmatrix} = A$$

Taking the ideal generated by the 9×9 minors of A, we obtain

$$\langle a_1 P, \ldots, a_{10} P \rangle,$$

where P is a certain homogeneous degree-eight polynomial with 1465 terms. Singular reveals that P is an irreducible polynomial, so that the resonance variety of M_6 consists of a single nonlinear component:

$$\mathcal{R}^1(M_6) = \{a \in \mathbb{C}^{10} \mid P = 0\}.$$

This is unlike the pure braid group and all earlier resonance variety calculations for arrangements, where the resonance had only linear components. Since $H^*(M_l) \subseteq H^*(M_n)$ for $l \leq n$, we conclude the following.

Proposition 5.6. *The set $\mathcal{R}^1(M_n) \subseteq H^1(M_n)$ contains a nonlinear component for all $n \geq 6$.*

Since the cohomology of $\pi_1(M_n)$ and M_n coincide, $\mathcal{R}^1(\pi_1(M_n))$ also contains a nonlinear component. The existence of this nonlinear component is connected to the topological property of formality.

5.1.1 Obstruction to Formality

A *differential graded algebra* (dg-algebra) is an anticommutative, graded $\mathbb{C}$-algebra $A = \bigoplus_{i \geq 0} A^i$ endowed with a cochain differential $d_A : A \to A$. A homomorphism between dg-algebras (A, d_A) and (B, d_B) is a map $f : A \to B$ which preserves degrees, i.e. $f(A^i) \subseteq B^i$ and satisfies $f(ab) = f(a)f(b)$ and $d_B f(a) = f(d_A a)$ for all $a, b \in A$. Such a homomorphism induces a homomorphism $\tilde{f} : H(A, d_A) \to H(B, d_B)$ of cohomology algebras. We call f a *quasi-isomorphism* if $\tilde{f}$ is an isomorphism.

Definition 5.7. A topological space is called **formal** if there is a zigzag of quasi-isomorphisms of dg-algebras

$$C^*(X) \xleftarrow{\simeq} \cdots \xrightarrow{\simeq} H^*(X),$$

where $C^*(X)$ is endowed with the standard differential and $H^*(X)$ is endowed with the zero differential.

A space is called q-formal if the zigzag of morphisms shown above induces isomorphisms in cohomology only up to degree q. Thus, X is formal iff it is q formal for all $q \geq 0$. In particular, if X is not 1-formal, then X is not formal.

There is a related notion of 1-formality for groups. A finitely generated group G is called 1-formal if its Malcev Lie algebra admits a quadratic presentation. It is known that the 1-formality of X concides with the 1-formality of $\pi_1(X)$. By the following proposition, $\pi_1(M_n)$ is not 1-formal for $n \geq 6$ since $\mathcal{R}^1(\pi_1(M_n))$ contains a nonlinear component. Thus, the space M_n is not 1-formal and hence not formal for $n \geq 6$. The nonformality of M_n was proved with other methods in [EHKR], and it demonstrates an essential difference between the pure cactus group and the pure braid group.

Proposition 5.8 ([DPS09])**.** *Let G be a finitely generated, 1-formal group. Then all irreducible components of $\mathcal{R}^1(G)$ are linear subspaces of $H^1(G)$.*

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