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POISSON APPROXIMATION OF RADEMACHER FUNCTIONALS BY THE CHEN-STEIN METHOD AND MALLIAVIN CALCULUS

KAI KROKOWSKI*

Abstract. New bounds on the total variation distance between the law of integer valued functionals of possibly non-symmetric and non-homogeneous infinite Rademacher sequences and the Poisson distribution are established. They are based on a combination of the Chen-Stein method and a discrete version of Malliavin calculus. We give some applications to shifted discrete multiple stochastic integrals.

1. Introduction

Stein's method and the Malliavin calculus have been combined for the first time by Nourdin and Peccati in the initial paper [9] in order to derive explicit bounds on the error in the normal and Gamma approximation of functionals of general Gaussian processes. This new approach to Stein's method, also known as the Malliavin-Stein method, has also been used to deduce quantitative central limit theorems for functionals of general Poisson measures (see [13]) and for functionals of infinite symmetric Rademacher sequences (see [11] and [6]). Here, the term symmetric Rademacher sequence refers to a sequence of independent and identically distributed random variables taking the values $+1$ and -1 with probability $1/2$ each.

The results in [11] and [6] are based on a product formula for multiple stochastic integrals (see Proposition 2.9 in [11]), whose proof relies on the simplicity of the underlying symmetric Rademacher sequence. The findings of [6] were further developed in [7], where a second order Poincaré type bound on the Kolmogorov distance between the law of functionals of possibly non-symmetric and non-homogeneous infinite Rademacher sequences and the standard normal distribution was derived. For analogues of such second order Poincaré type inequalities in the Gaussian and Poisson case see [10] and [8], respectively. One advantage of the bound in [7] is that it can be further evaluated without the need of a product formula for multiple stochastic integrals.

Poisson approximation by a combination of the Chen-Stein method and Malliavin calculus has first been tackled in [12], where the author computed explicit

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bounds on the total variation distance between the law of integer valued functionals of general Poisson measures and a Poisson distribution. Furthermore, sufficient conditions for the convergence in distribution of suitably shifted multiple stochastic integrals to a Poisson random variable and rates of convergence for the Poisson approximation of statistics associated with geometric random graphs were covered. For further works in the framework of the Chen-Stein method and Malliavin calculus see, e.g., [3], [16] and [17].

The purpose of this paper is to combine the Chen-Stein method and a discrete version of Malliavin calculus (as developed in [14]), and thus, to continue the work of [6] and [7] to the case of Poisson approximation. A general bound on the total variation distance between the law of integer valued functionals of possibly non-symmetric and non-homogeneous infinite Rademacher sequences and the Poisson distribution is shown (see Theorem 3.1). Applications to shifted multiple stochastic integrals are considered (see Theorem 3.4 and Corollary 3.5 as well as Theorem 3.7 and Corollary 3.9). For this, a generalization of the product formula from [11] to multiple stochastic integrals based on an underlying possibly non-symmetric and non-homogeneous Rademacher sequence is proved (see Proposition 2.2). In addition, using the techniques provided in [7], a second order Poincaré type inequality is deduced from the general bound (see Theorem 3.13).

The remainder of this paper is built up as follows. Section 2 collects the bases of the discrete Malliavin calculus as well as the product formula for multiple stochastic integrals. Furthermore, a short introduction to the Chen-Stein method is given. Section 3 contains all of the main results and their proofs. Section 4 serves as appendix and bears the proof of the product formula and, additionally, a standard approximation argument that is used within some of the proofs in this paper.

The authors of [15] have also developed bounds on the total variation distance between the law of integer valued functionals of possibly non-symmetric and non-homogeneous infinite Rademacher sequences and the Poisson distribution by using a generalization of the product formula for multiple stochastic integrals in [11] as well. In particular, Theorem 3.1 and Corollary 3.3 here are related to Theorem 6.3 in [15], Theorem 3.4 and Corollary 3.5 are related to Theorem 7.1 in [15], and Theorem 3.7 and Remark 3.8 are related to Theorem 8.2 and Proposition 8.3, respectively, in [15]. However, the corresponding results of [15] and this paper were worked out independently of each other and differ (see, e.g., Remark 3.6). In addition, we contribute a second order Poincaré type bound which is not provided in [15].

2. Preliminaries

2.1. Rademacher sequences. Let $p := (p_k)_{k\in\mathbb{N}}$ be a sequence of success probabilities fulfilling $0 < p_k < 1$, for every $k \in \mathbb{N}$, and let $q := (q_k)_{k\in\mathbb{N}}$ be the corresponding sequence of failure probabilities with $q_k := 1 - p_k$, for every $k \in \mathbb{N}$. Furthermore, let $(\Omega, \mathcal{F}, P)$ be a probability space with $\Omega := \{-1, +1\}^{\mathbb{N}}$, $\mathcal{F} := \mathcal{P}(\{-1, +1\})^{\otimes \mathbb{N}}$ and $P := \bigotimes_{k=1}^{\infty}(p_k\delta_{+1} + q_k\delta_{-1})$. Now, let $X := (X_k)_{k\in\mathbb{N}}$ be a sequence of independent random variables defined on $(\Omega, \mathcal{F}, P)$ by $X_k(\omega) := \omega_k$, for every $k \in \mathbb{N}$ and $\omega := (\omega_k)_{k\in\mathbb{N}} \in \Omega$. Here, we refer to the sequence X as (possibly non-symmetric and non-homogeneous) Rademacher sequence. In the following, we will introduce discrete multiple stochastic integrals on the basis of our Rademacher sequence. To this end, we also define the standardized sequence $Y := (Y_k)_{k\in\mathbb{N}}$ with

$$Y_k := (X_k - \mathbb{E}[X_k])/\sqrt{\operatorname{Var}(X_k)} = (X_k - p_k + q_k)/(2\sqrt{p_k q_k}),$$

for every $k \in \mathbb{N}$.

2.2. Kernels and contractions. Let κ be the counting measure on $\mathbb{N}$. We put $\ell^2(\mathbb{N})^{\otimes n} := L^2(\mathbb{N}^n, \mathcal{P}(\mathbb{N})^{\otimes n}, \kappa^{\otimes n})$, for every $n \in \mathbb{N}$. In the following, we refer to the elements of $\ell^2(\mathbb{N})^{\otimes n}$ as kernels. Let $\ell^2(\mathbb{N})^{\circ n}$ denote the subset of $\ell^2(\mathbb{N})^{\otimes n}$ of symmetric kernels. Furthermore, let $\ell_0^2(\mathbb{N})^{\otimes n}$ denote the subset of kernels vanishing on diagonals, i.e. vanishing on the complement of the set $\Delta_n := \{(i_1, \ldots, i_n) \in \mathbb{N}^n : i_j \neq i_k \text{ for } j \neq k\}$. We then put $\ell_0^2(\mathbb{N})^{\circ n} := \ell^2(\mathbb{N})^{\circ n} \cap \ell_0^2(\mathbb{N})^{\otimes n}$. For $n, m \in \mathbb{N}$, take two kernels $f \in \ell_0^2(\mathbb{N})^{\circ n}$ and $g \in \ell_0^2(\mathbb{N})^{\circ m}$. Now, for $r = 0, \ldots, n \wedge m$ and $\ell = 0, \ldots, r$, the contraction of f and g is defined by

$$\begin{aligned} &f \star_r^\ell g(i_1, \ldots, i_{n-r}, k_1, \ldots, k_{r-\ell}, j_1, \ldots, j_{m-r}) \\ &:= \sum_{(a_1,\ldots,a_\ell)\in\Delta_\ell} f(i_1, \ldots, i_{n-r}, k_1, \ldots, k_{r-\ell}, a_1, \ldots, a_\ell) \\ &\qquad\qquad \times g(j_1, \ldots, j_{m-r}, k_1, \ldots, k_{r-\ell}, a_1, \ldots, a_\ell), \end{aligned}$$

that is, by identifying r of the n variables of f with r of the m variables of g and then integrating out ℓ of the r identified variables with respect to the counting measure κ. Note that $f \star_r^\ell g \in \ell^2(\mathbb{N})^{\otimes n+m-r-\ell}$, since $\|f \star_r^\ell g\|_{\ell^2(\mathbb{N})^{\otimes n+m-r-\ell}} \leq \|f\|_{\ell^2(\mathbb{N})^{\otimes n}} \|g\|_{\ell^2(\mathbb{N})^{\otimes m}}$ (cf. Lemma 2.4 in [11]). Even though $f \in \ell_0^2(\mathbb{N})^{\circ n}$ and $g \in \ell_0^2(\mathbb{N})^{\circ m}$, the contraction $f \star_r^\ell g$ must neither be symmetric nor be vanishing on diagonals. Therefore, we define the canonical symmetrization of a function f on $\mathbb{N}^n$ by $\widetilde{f}(i_1, \ldots, i_n) := \frac{1}{n!}\sum_\sigma f(i_{\sigma(1)}, \ldots, i_{\sigma(n)})$, where the sum runs over all permutations σ of the set $\{1, \ldots, n\}$. Note that, if $f \in \ell^2(\mathbb{N})^{\otimes n}$, then $\widetilde{f} \in \ell^2(\mathbb{N})^{\otimes n}$, since $\|\widetilde{f}\|_{\ell^2(\mathbb{N})^{\otimes n}} \leq \|f\|_{\ell^2(\mathbb{N})^{\otimes n}}$.

2.3. Discrete multiple stochastic integrals and chaos representation. For $n \in \mathbb{N}$ and $f \in \ell_0^2(\mathbb{N})^{\circ n}$, we define the discrete multiple stochastic integral of order n of f by

$$\begin{aligned} J_n(f) &:= \sum_{(i_1,\ldots,i_n)\in\mathbb{N}^n} f(i_1, \ldots, i_n) Y_{i_1} \cdot \ldots \cdot Y_{i_n} \\ &= \sum_{(i_1,\ldots,i_n)\in\Delta_n} f(i_1, \ldots, i_n) Y_{i_1} \cdot \ldots \cdot Y_{i_n} \\ &= n! \sum_{1\leq i_1<\ldots<i_n<\infty} f(i_1, \ldots, i_n) Y_{i_1} \cdot \ldots \cdot Y_{i_n}. \end{aligned} \tag{2.1}$$

In addition, we put $\ell^2(\mathbb{N})^{\otimes 0} := \mathbb{R}$ and $J_0(c) := c$, for every $c \in \mathbb{R}$.

For every $n \in \mathbb{N}$, the subspace $\{J_n(f) : f \in \ell_0^2(\mathbb{N})^{\circ n}\}$ of $L^2(\Omega)$ is called the Rademacher chaos of order n. Now, every square-integrable Rademacher functional $F \in L^2(\Omega)$ admits a unique decomposition of the form

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} J_n(f_n) \tag{2.2}$$

with $f_n \in \ell_0^2(\mathbb{N})^{\circ n}$, for every $n \in \mathbb{N}$ (cf. Proposition 6.7 in [14]). We call (2.2) the chaos representation of F, where the series converges in $L^2(\Omega)$.

We will now prepare for the presentation of a product formula for discrete stochastic integrals. The following observation is crucial to derive such a product formula (cf. Chapter 5 in [14]).

Lemma 2.1. *For every $k \in \mathbb{N}$, Y_k^2 admits the chaos representation*

$$Y_k^2 = 1 + \varphi_k Y_k, \tag{2.3}$$

where the sequence $\varphi := (\varphi_k)_{k\in\mathbb{N}}$ is defined by $\varphi_k := (q_k - p_k)/\sqrt{p_k q_k}$, for every $k \in \mathbb{N}$.

For $n, m \in \mathbb{N}$, take two kernels $f \in \ell_0^2(\mathbb{N})^{\circ n}$ and $g \in \ell_0^2(\mathbb{N})^{\circ m}$. Now, for $r = 1, \ldots, n \wedge m$ and $\ell = 0, \ldots, r-1$, we define the weighted contraction of f and g by

$$\begin{aligned} &\varphi^{*r-\ell}(f \star_r^\ell g)(i_1, \ldots, i_{n-r}, k_1, \ldots, k_{r-\ell}, j_1, \ldots, j_{m-r}) \\ &:= \varphi_{k_1} \cdot \ldots \cdot \varphi_{k_{r-\ell}} f \star_r^\ell g(i_1, \ldots, i_{n-r}, k_1, \ldots, k_{r-\ell}, j_1, \ldots, j_{m-r}). \end{aligned}$$

Note that the indices $k_1, \ldots, k_{r-\ell}$ of the factors in the product $\varphi_{k_1} \cdot \ldots \cdot \varphi_{k_{r-\ell}}$ are the $r - \ell$ variables of the contraction $f \star_r^\ell g$ that are identified but not integrated out. For $r = 0, \ldots, n \wedge m$, we further define

$$\varphi^{*0}(f \star_r^r g)(i_1, \ldots, i_{n-r}, j_1, \ldots, j_{m-r}) := f \star_r^r g(i_1, \ldots, i_{n-r}, j_1, \ldots, j_{m-r}).$$

Now, the following proposition states a formula for the product of discrete multiple stochastic integrals. Note that this is a generalization of Proposition 2.9 in [11] to the case of stochastic integrals based on a possibly non-symmetric and non-homogeneous infinite Rademacher sequence. We refer to the appendix for a proof of the statement. Also note that the following Proposition 2.2 corresponds to Proposition 5.1 in [15].

Proposition 2.2 (Product formula)**.** *Let $n, m \in \mathbb{N}$ and $f \in \ell_0^2(\mathbb{N})^{\circ n}, g \in \ell_0^2(\mathbb{N})^{\circ m}$. Furthermore, let $(\widetilde{\varphi^{*r-\ell}(f \star_r^\ell g)})\,\mathbb{1}_{\Delta_{n+m-r-\ell}} \in \ell_0^2(\mathbb{N})^{\circ n+m-r-\ell}$, for $r = 1, \ldots, n\wedge m$ and $\ell = 0, \ldots, r-1$. Then,*

$$J_n(f)J_m(g) = \sum_{r=0}^{n\wedge m} r! \binom{n}{r}\binom{m}{r} \sum_{\ell=0}^{r} \binom{r}{\ell} J_{n+m-r-\ell}\Big((\widetilde{\varphi^{*r-\ell}(f \star_r^\ell g)})\,\mathbb{1}_{\Delta_{n+m-r-\ell}} \Big) \tag{2.4}$$

$$= \sum_{r=0}^{n\wedge m} r! \binom{n}{r}\binom{m}{r} J_{n+m-2r}\left((\widetilde{f \star_r^r g})\,\mathbb{1}_{\Delta_{n+m-2r}} \right)$$

$$+\sum_{r=1}^{n\wedge m} r!\binom{n}{r}\binom{m}{r}\sum_{\ell=0}^{r-1}\binom{r}{\ell}J_{n+m-r-\ell}\Big((\varphi^{*r-\ell}\widetilde{(f\star_r^\ell g))}\,\mathbb{1}_{\Delta_{n+m-r-\ell}}\Big), \tag{2.5}$$

where we put $\mathbb{1}_{\Delta_0} := 1$.

Remark 2.3.

(i) In Proposition 2.2, sufficient conditions for $(\varphi^{*r-\ell}\widetilde{(f\star_r^\ell g))}\,\mathbb{1}_{\Delta_{n+m-r-\ell}}$ to be an element of $\ell^2(\mathbb{N})^{\otimes n+m-r-\ell}$, for every $r = 1,\dots,n\wedge m$ and $\ell = 0,\dots,r-1$, are given, e.g., if the sequence φ is either constant or fulfills $\|\varphi\|_{\ell^2(\mathbb{N})} < \infty$.

(ii) While we will use (2.4) in further applications, the representation of the product formula in (2.5) exhibits the relation between the general case of a possibly non-symmetric and non-homogeneous Rademacher sequence and the case of a symmetric Rademacher sequence. In the case of a symmetric Rademacher sequence X, i.e. $p_k = q_k = 1/2$, for every $k\in\mathbb{N}$, the coefficients φ_k of the chaos representation of Y_k^2 in (2.3) vanish, for every $k\in\mathbb{N}$, so that Proposition 2.2 reproduces Proposition 2.9 in [11].

The next corollary states an isometry formula for stochastic integrals as seen in Proposition 4.2 in [14]. Note that this is also an immediate conclusion from the product formula in Proposition 2.2, since, for every $n\in\mathbb{N}$ and $f\in\ell_0^2(\mathbb{N})^{\circ n}$, $\mathbb{E}[J_n(f)] = 0$.

Corollary 2.4 (Isometry formula)**.** *Let* $n,m\in\mathbb{N}$ *and* $f\in\ell_0^2(\mathbb{N})^{\circ n}, g\in\ell_0^2(\mathbb{N})^{\circ m}$. *Then,*

$$\mathbb{E}[J_n(f)J_m(g)] = \begin{cases} n!\langle f,g\rangle_{\ell^2(\mathbb{N})^{\otimes n}}, & \text{if } n = m,\\ 0, & \text{if } n\neq m.\end{cases} \tag{2.6}$$

2.4. Discrete Malliavin calculus. We define the discrete gradient operator D. For every $\omega = (\omega_1,\omega_2,\dots)\in\Omega$ and $k\in\mathbb{N}$, let $\omega_+^k := (\omega_1,\dots,\omega_{k-1},+1,\omega_{k+1},\dots)$ and $\omega_-^k := (\omega_1,\dots,\omega_{k-1},-1,\omega_{k+1},\dots)$. Furthermore, for every $F\in L^1(\Omega)$, $\omega\in\Omega$ and $k\in\mathbb{N}$, let $F_k^+(\omega) := F(\omega_+^k)$ and $F_k^-(\omega) := F(\omega_-^k)$. For $F\in L^1(\Omega)$, the discrete gradient operator is defined by $DF := (D_kF)_{k\in\mathbb{N}}$ with

$$D_kF := \sqrt{p_kq_k}(F_k^+ - F_k^-), \tag{2.7}$$

for every $k\in\mathbb{N}$. Note that it immediately follows from (2.7) that, for every $k\in\mathbb{N}$, D_kF is independent of X_k. Now, let $F\in L^2(\Omega)$ have the chaos representation $F = \mathbb{E}[F] + \sum_{n=1}^\infty J_n(f_n)$ with kernels $f_n\in\ell_0^2(\mathbb{N})^{\circ n}$, for every $n\in\mathbb{N}$. Then, for every $k\in\mathbb{N}$, $D_kF\in L^2(\Omega)$ and has the chaos representation

$$D_kF = \sum_{n=1}^\infty nJ_{n-1}(f_n(\,\cdot\,,k)),$$

where, for every $n\in\mathbb{N}$, $f_n(\,\cdot\,,k)\in\ell_0^2(\mathbb{N})^{\circ n-1}$ denotes the kernel f_n with one of its components fixed, thus as a function in only $n-1$ variables (cf. Chapter 2.3 in [7]). In addition, for $F\in L^1(\Omega)$ and $m\in\mathbb{N}$, the iterated discrete gradient operator of order m is defined by $D^mF := (D^m_{k_1,\dots,k_m}F)_{k_1,\dots,k_m\in\mathbb{N}}$ with $D^m_{k_1,\dots,k_m}F := D_{k_m}(D^{m-1}_{k_1,\dots,k_{m-1}}F)$, for every $k_1,\dots,k_m\in\mathbb{N}$, where we put $D^0_{k_1,\dots,k_0}F := F$.

Given $F \in L^2(\Omega)$ with chaos representation $F = \mathbb{E}[F] + \sum_{n=1}^{\infty} J_n(f_n)$ as above and $m \in \mathbb{N}$, we say $F \in \operatorname{dom}(D^m)$, if

$$\mathbb{E}[\|D^m F\|^2_{\ell^2(\mathbb{N})^{\otimes m}}] = \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} n! \|f_n\|^2_{\ell^2(\mathbb{N})^{\otimes n}} < \infty. \tag{2.8}$$

We will now define the discrete divergence operator δ and its domain $\operatorname{dom}(\delta)$. For $n \in \mathbb{N}$ and $f_n \in \ell_0^2(\mathbb{N})^{\circ n-1} \otimes \ell^2(\mathbb{N})$ we consider the sequence $u := (u_k)_{k\in\mathbb{N}}$ with $u_k := \sum_{n=1}^{\infty} J_{n-1}(f_n(\,\cdot\,, k))$, for every $k \in \mathbb{N}$. For such a sequence u, we say $u \in \operatorname{dom}(\delta)$, if

$$\sum_{n=1}^{\infty} n! \|\widetilde{f}_n \mathbb{1}_{\Delta_n}\|^2_{\ell^2(\mathbb{N})^{\otimes n}} < \infty. \tag{2.9}$$

For $u \in \operatorname{dom}(\delta)$, the discrete divergence operator δ is then defined by

$$\delta(u) := \sum_{n=1}^{\infty} J_n(\widetilde{f}_n \mathbb{1}_{\Delta_n}).$$

Note that (2.9) is equivalent to $\mathbb{E}[(\delta(u))^2] < \infty$. Now, δ is the adjoint of D (cf. Proposition 9.2 in [14]).

Lemma 2.5. *Let $F \in \operatorname{dom}(D)$ and $u \in \operatorname{dom}(\delta)$. Then,*

$$\mathbb{E}[F\delta(u)] = \mathbb{E}[\langle DF, u\rangle_{\ell^2(\mathbb{N})}]. \tag{2.10}$$

Next, we define the discrete Ornstein-Uhlenbeck operator L and its (pseudo-)inverse L^{-1}. Given $F \in L^2(\Omega)$, again with chaos representation $F = \mathbb{E}[F] + \sum_{n=1}^{\infty} J_n(f_n)$ as above, we say $F \in \operatorname{dom}(L)$, if

$$\sum_{n=1}^{\infty} n^2 n! \|f_n\|^2_{\ell^2(\mathbb{N})^{\otimes n}} < \infty.$$

For $F \in \operatorname{dom}(L)$, the discrete Ornstein-Uhlenbeck operator L is then defined by

$$LF := -\sum_{n=1}^{\infty} n J_n(f_n).$$

For centered $F \in L^2(\Omega)$, its (pseudo-)inverse is defined by

$$L^{-1}F := -\sum_{n=1}^{\infty} \frac{1}{n} J_n(f_n).$$

The following lemma states the relation between the operators D, δ and L (cf. Chapter 10 in [14]).

Lemma 2.6. *It holds that*

$$L = -\delta D. \tag{2.11}$$

Finally, we present an integration by parts formula, which is one of the main contributions to the discrete Malliavin-Stein method.

Lemma 2.7 (Integration by parts formula)**.** *Let $F, G \in \operatorname{dom}(D)$. Then,*

$$\mathbb{E}[(F - \mathbb{E}[F])G] = \mathbb{E}[\langle -DL^{-1}(F - \mathbb{E}[F]), DG\rangle_{\ell^2(\mathbb{N})}]. \tag{2.12}$$

Proof. Relation (2.11) and the adjointness of D and δ in (2.10) yield

$$\begin{aligned}&\mathbb{E}[(F-\mathbb{E}[F])G]=\mathbb{E}[LL^{-1}(F-\mathbb{E}[F])G]\\&=\mathbb{E}[-\delta DL^{-1}(F-\mathbb{E}[F])G]\\&=\mathbb{E}[\langle -DL^{-1}(F-\mathbb{E}[F]),DG\rangle_{\ell^2(\mathbb{N})}].\end{aligned}$$

□

2.5. The Chen-Stein method. Stein's method for Poisson approximation, also known as the Chen-Stein method, has been introduced by Chen in [4]. Since then, the method was further developed by Barbour and others, see, e.g., [1]. The starting point of the method is the following characterization of a Poisson distribution. A random variable Z has a Poisson distribution with mean $\lambda > 0$, if and only if, for every bounded function $f : \mathbb{N}_0 := \mathbb{N} \cup \{0\} \to \mathbb{R}$,

$$\mathbb{E}[\lambda f(Z+1) - Zf(Z)] = 0.$$

Now, the main idea is to set the total variation distance between the law of a given random variable and a Poisson distribution in relation to the characterization above. The link to do so is given by the Chen-Stein equation. To state the equation, let $\mathrm{Po}(\lambda)$ be a Poisson random variable with mean $\lambda > 0$. Then, for every $A \subseteq \mathbb{N}_0$ and $k \in \mathbb{N}_0$, the Chen-Stein equation is given by

$$\lambda f(k+1) - kf(k) = \mathbb{1}_{\{k\in A\}} - P(\mathrm{Po}(\lambda) \in A). \tag{2.13}$$

For $k \in \mathbb{N}$, (2.13) has a unique and bounded solution $f_{\lambda,A} : \mathbb{N} \to \mathbb{R}$ with

$$f_{\lambda,A}(k) := \frac{(k-1)!}{\lambda^k}\sum_{j=0}^{k-1}(\mathbb{1}_{\{j\in A\}} - P(\mathrm{Po}(\lambda)\in A))\frac{\lambda^j}{j!}. \tag{2.14}$$

Since, for $k = 0$, the value of $f(0)$ does not contribute to (2.13), we conventionally put $f_{\lambda,A}(0) = 0$. Given a function $f : \mathbb{N}_0 \to \mathbb{R}$, we define the forward difference of f by $\Delta f(k) := f(k+1) - f(k)$, for every $k \in \mathbb{N}_0$. Furthermore, we define the iterated forward difference of f by $\Delta^2 f(k) := \Delta(\Delta f(k))$, for every $k \in \mathbb{N}_0$. Moreover, the supremum norm of f is given by $\|f\|_\infty := \sup_{k\in\mathbb{N}_0} |f(k)|$. The following bounds hold for the solution of the Chen-Stein equation in (2.14) (cf. Lemma 1.1.1 and Remark 1.1.2 in [1]):

$$\|f_{\lambda,A}\|_\infty \le 1 \wedge \sqrt{\frac{2}{e\lambda}}, \quad \|\Delta f_{\lambda,A}\|_\infty \le \frac{1-e^{-\lambda}}{\lambda}. \tag{2.15}$$

In addition, the relation $\|\Delta^2 f_{\lambda,A}\|_\infty \le 2\|\Delta f_{\lambda,A}\|_\infty$ gives the obvious bound

$$\|\Delta^2 f_{\lambda,A}\|_\infty \le \frac{2(1-e^{-\lambda})}{\lambda}. \tag{2.16}$$

Note that the bound $\|\Delta^2 f_{\lambda,A}\|_\infty \le 2(1-e^{-\lambda})/\lambda^2$ does not follow from Theorem 1.3 in [5] as stated in [12] and [15]. However, Theorem 1.3 in [5] does lead to a bound $\|\Delta^2 f_{\lambda,A}\|_\infty \le 2/\lambda$.

3. Main Results

In the following, we will deduce a bound on the error in the Poisson approximation of general integer valued functionals of possibly non-symmetric and non-homogeneous infinite Rademacher sequences with respect to the total variation distance. The total variation distance between the distributions of two random variables X and Y with values in $\mathbb{N}_0$ is defined by

$$d_{TV}(X,Y) := \sup_{A\subseteq\mathbb{N}_0} |P(X\in A) - P(Y\in A)|.$$

For a corresponding bound on the error in the Poisson approximation of integer valued functionals of general Poisson measures see Theorem 3.1 in [12]. Again, note that the following Theorem 3.1 and Corollary 3.3 are related to Theorem 6.3 in [15]

Theorem 3.1. *Let $F \in \operatorname{dom}(D)$ with values in $\mathbb{N}_0$ and let $\operatorname{Po}(\lambda)$ be a Poisson random variable with mean $\lambda > 0$. Then,*

$$\begin{aligned} &d_{TV}(F,\operatorname{Po}(\lambda)) \\ &\leq \Big(1\wedge\sqrt{\frac{2}{e\lambda}}\Big)|\lambda-\mathbb{E}[F]| + \frac{1-e^{-\lambda}}{\lambda}\,\mathbb{E}[|\lambda - \langle DF, -DL^{-1}(F-\mathbb{E}[F])\rangle_{\ell^2(\mathbb{N})}|] \\ &\quad + \frac{1-e^{-\lambda}}{\lambda}\,\mathbb{E}\Big[\Big\langle \frac{1}{\sqrt{pq}}DF(DF+\sqrt{pq}X), |-DL^{-1}(F-\mathbb{E}[F])|\Big\rangle_{\ell^2(\mathbb{N})}\Big]. \qquad (3.1)\end{aligned}$$

Proof. By the Chen-Stein equation in (2.13) and the integration by parts formula in (2.12), we have, for every $A\subseteq\mathbb{N}_0$,

$$\begin{aligned} &P(F\in A) - P(\operatorname{Po}(\lambda)\in A) = \mathbb{E}[\lambda f_{\lambda,A}(F+1)] - \mathbb{E}[Ff_{\lambda,A}(F)] \\ &= \mathbb{E}[\lambda(f_{\lambda,A}(F+1)-f_{\lambda,A}(F))] - \mathbb{E}[(F-\mathbb{E}[F])f_{\lambda,A}(F)] - \mathbb{E}[(\mathbb{E}[F]-\lambda)f_{\lambda,A}(F)] \\ &= \mathbb{E}[\lambda\Delta f_{\lambda,A}(F)] - \mathbb{E}[\langle Df_{\lambda,A}(F), -DL^{-1}(F-\mathbb{E}[F])\rangle_{\ell^2(\mathbb{N})}] \\ &\quad - \mathbb{E}[(\mathbb{E}[F]-\lambda)f_{\lambda,A}(F)]. \qquad (3.2)\end{aligned}$$

We will now further deduce $Df_{\lambda,A}(F)$. For every $k\in\mathbb{N}$, we have

$$\begin{aligned} &D_kf_{\lambda,A}(F) = \sqrt{p_kq_k}(f_{\lambda,A}(F_k^+) - f_{\lambda,A}(F_k^-)) \\ &= \Delta f_{\lambda,A}(F)\cdot D_kF + \sqrt{p_kq_k}(f_{\lambda,A}(F_k^+) - f_{\lambda,A}(F_k^-) - \Delta f_{\lambda,A}(F)(F_k^+ - F_k^-)) \\ &= \Delta f_{\lambda,A}(F)\cdot D_kF + R_k(F) \qquad (3.3)\end{aligned}$$

with

$$R_k(F) := \sqrt{p_kq_k}(f_{\lambda,A}(F_k^+) - f_{\lambda,A}(F_k^-) - \Delta f_{\lambda,A}(F)(F_k^+ - F_k^-)).$$

Now, let $a,k\in\mathbb{N}_0$ with $k\geq a+2$. Then,

$$f_{\lambda,A}(k) - f_{\lambda,A}(a) - \Delta f_{\lambda,A}(a)(k-a) = \sum_{j=1}^{k-a-1} j\cdot\Delta^2 f_{\lambda,A}(k-j-1).$$

Similarly, for every $a,k\in\mathbb{N}_0$ with $k\leq a-1$, one gets

$$f_{\lambda,A}(k) - f_{\lambda,A}(a) - \Delta f_{\lambda,A}(a)(k-a) = \sum_{j=1}^{a-k} j\cdot\Delta^2 f_{\lambda,A}(k+j-1).$$

Moreover, for every $a \in \mathbb{N}_0$ and $k \in \{a, a+1\}$, it holds that

$$f_{\lambda,A}(k) - f_{\lambda,A}(a) - \Delta f_{\lambda,A}(a)(k-a) = 0.$$

Thus, for every $a, k \in \mathbb{N}_0$,

$$|f_{\lambda,A}(k) - f_{\lambda,A}(a) - \Delta f_{\lambda,A}(a)(k-a)| \le \frac{\|\Delta^2 f_{\lambda,A}\|_\infty}{2}(k-a)(k-a-1). \quad (3.4)$$

We will now use (3.4) to further estimate the error term $R_k(F)$ in the chain rule at (3.3). Note that, for every $k \in \mathbb{N}$, we have

$$\begin{aligned} R_k(F) = &\sqrt{p_k q_k}(f_{\lambda,A}(F_k^+) - f_{\lambda,A}(F_k^-) - \Delta f_{\lambda,A}(F_k^-)(F_k^+ - F_k^-))\, \mathbb{1}_{\{X_k=-1\}} \\ &- \sqrt{p_k q_k}(f_{\lambda,A}(F_k^-) - f_{\lambda,A}(F_k^+) - \Delta f_{\lambda,A}(F_k^+)(F_k^- - F_k^+))\, \mathbb{1}_{\{X_k=+1\}}\,. \end{aligned}$$

It then follows by (3.4) that, for every $k \in \mathbb{N}$,

$$\begin{aligned} &|f_{\lambda,A}(F_k^+) - f_{\lambda,A}(F_k^-) - \Delta f_{\lambda,A}(F_k^-)(F_k^+ - F_k^-)| \\ &\le \frac{\|\Delta^2 f_{\lambda,A}\|_\infty}{2}(F_k^+ - F_k^-)(F_k^+ - F_k^- - 1) \end{aligned}$$

and

$$\begin{aligned} &|f_{\lambda,A}(F_k^-) - f_{\lambda,A}(F_k^+) - \Delta f_{\lambda,A}(F_k^+)(F_k^- - F_k^+)| \\ &\le \frac{\|\Delta^2 f_{\lambda,A}\|_\infty}{2}(F_k^+ - F_k^-)(F_k^+ - F_k^- + 1). \end{aligned}$$

Thus, for every $k \in \mathbb{N}$,

$$\begin{aligned} |R_k(F)| &\le \frac{\|\Delta^2 f_{\lambda,A}\|_\infty}{2}\sqrt{p_k q_k}(F_k^+ - F_k^-)(F_k^+ - F_k^- - 1)\, \mathbb{1}_{\{X_k=-1\}} \\ &\quad + \frac{\|\Delta^2 f_{\lambda,A}\|_\infty}{2}\sqrt{p_k q_k}(F_k^+ - F_k^-)(F_k^+ - F_k^- + 1)\, \mathbb{1}_{\{X_k=+1\}} \\ &= \frac{\|\Delta^2 f_{\lambda,A}\|_\infty}{2}\sqrt{p_k q_k}(F_k^+ - F_k^-)(F_k^+ - F_k^- + X_k) \\ &= \|\Delta^2 f_{\lambda,A}\|_\infty \frac{1}{2\sqrt{p_k q_k}} D_k F(D_k F + \sqrt{p_k q_k} X_k). \end{aligned} \quad (3.5)$$

Putting $R(F) := (R_k(F))_{k\in\mathbb{N}}$, we then deduce from (3.2) by (3.3) and (3.5) that, for every $A \subseteq \mathbb{N}_0$,

$$\begin{aligned} &|P(F \in A) - P(\mathrm{Po}(\lambda) \in A)| \\ &= |\mathbb{E}[\Delta f_{\lambda,A}(F)(\lambda - \langle DF, -DL^{-1}(F - \mathbb{E}[F])\rangle_{\ell^2(\mathbb{N})}] \\ &\quad - \mathbb{E}[\langle R(F), -DL^{-1}(F - \mathbb{E}[F])\rangle_{\ell^2(\mathbb{N})})] - \mathbb{E}[(\mathbb{E}[F] - \lambda) f_{\lambda,A}(F)]| \\ &\le \|\Delta f_{\lambda,A}\|_\infty\, \mathbb{E}[|\lambda - \langle DF, -DL^{-1}(F - \mathbb{E}[F])\rangle_{\ell^2(\mathbb{N})}|] \\ &\quad + \|\Delta^2 f_{\lambda,A}\|_\infty\, \mathbb{E}\left[\left\langle \frac{1}{2\sqrt{pq}} DF(DF + \sqrt{pq}X), |-DL^{-1}(F - \mathbb{E}[F])| \right\rangle_{\ell^2(\mathbb{N})}\right] \\ &\quad + \|f_{\lambda,A}\|_\infty |\lambda - \mathbb{E}[F]|. \end{aligned}$$

(3.1) now follows by (2.15) and (2.16). □

Remark 3.2. Note that the arguments used in the proof of Theorem 3.1 are not restricted to the choice of the total variation distance to measure the distance between the laws of F and $\mathrm{Po}(\lambda)$. Indeed, choosing any arbitrary class $\mathcal{H}$ of bounded test functions $h : \mathbb{N}_0 \to \mathbb{R}$ would lead to a bound

$$\begin{aligned}&\sup_{h\in\mathcal{H}} |\mathbb{E}[h(F)] - \mathbb{E}[h(\mathrm{Po}(\lambda))]|\\ &\le \|f_h\|_\infty |\lambda - \mathbb{E}[F]| + \|\Delta f_h\|_\infty \,\mathbb{E}[|\lambda - \langle DF, -DL^{-1}(F - \mathbb{E}[F])\rangle_{\ell^2(\mathbb{N})}|]\\ &\quad + \|\Delta^2 f_h\|_\infty \,\mathbb{E}\Big[\Big\langle \frac{1}{2\sqrt{pq}} DF(DF + \sqrt{pq}X), |-DL^{-1}(F - \mathbb{E}[F])|\Big\rangle_{\ell^2(\mathbb{N})}\Big],\end{aligned}$$

where f_h denotes the solution to the corresponding Chen-Stein equation. Taking, e.g., $\mathcal{H}$ as the set of all Lipschitz functions on $\mathbb{N}_0$ with Lipschitz constant not greater than 1 yields the following bound on the Wasserstein distance:

$$\begin{aligned}&d_W(F, \mathrm{Po}(\lambda))\\ &\le |\lambda - \mathbb{E}[F]| + \Big(1 \wedge \frac{8}{3\sqrt{2e\lambda}}\Big)\,\mathbb{E}[|\lambda - \langle DF, -DL^{-1}(F - \mathbb{E}[F])\rangle_{\ell^2(\mathbb{N})}|]\\ &\quad + \Big(\frac{4}{3} \wedge \frac{2}{\lambda}\Big)\,\mathbb{E}\Big[\Big\langle \frac{1}{2\sqrt{pq}} DF(DF + \sqrt{pq}X), |-DL^{-1}(F - \mathbb{E}[F])|\Big\rangle_{\ell^2(\mathbb{N})}\Big],\end{aligned}$$

where we took the bounds for $\|f_h\|_\infty, \|\Delta f_h\|_\infty$ and $\|\Delta^2 f_h\|_\infty$ from Theorem 1.1 in [2].

The following corollary shows that we can rewrite the bound in (3.1) without resorting to the Rademacher sequence X. In this way, our bound here gets a representation closer to the one of the bound in Theorem 3.1 in [12].

Corollary 3.3. *Let $F \in \mathrm{dom}(D)$ with values in $\mathbb{N}_0$ and let $\mathrm{Po}(\lambda)$ be a Poisson random variable with mean $\lambda > 0$. Then,*

$$\begin{aligned}&d_{TV}(F, \mathrm{Po}(\lambda))\\ &\le \Big(1 \wedge \sqrt{\frac{2}{e\lambda}}\Big)|\lambda - \mathbb{E}[F]| + \frac{1 - e^{-\lambda}}{\lambda}\,\mathbb{E}[|\lambda - \langle DF, -DL^{-1}(F - \mathbb{E}[F])\rangle_{\ell^2(\mathbb{N})}|]\\ &\quad + \frac{1 - e^{-\lambda}}{\lambda}\sum_{k=1}^{\infty} \frac{1}{\sqrt{p_k q_k}}\,\mathbb{E}[D_kF(D_kF + \sqrt{p_k q_k}(p_k - q_k)) \cdot |-D_kL^{-1}(F - \mathbb{E}[F])|].\end{aligned}$$

Proof. We only have to consider the last summand of the bound in (3.1) separately. Since, for every $k \in \mathbb{N}$, $D_kF(D_kF + \sqrt{p_kq_k}X_k) \ge 0$ by (3.5) and D_kF is independent of X_k for every $F \in L^1(\Omega)$, we get

$$\begin{aligned}&\mathbb{E}\Big[\Big\langle \frac{1}{\sqrt{pq}} DF(DF + \sqrt{pq}X), |-DL^{-1}(F - \mathbb{E}[F])|\Big\rangle_{\ell^2(\mathbb{N})}\Big]\\ &= \sum_{k=1}^{\infty} \frac{1}{\sqrt{p_k q_k}}\,\mathbb{E}[D_kF(D_kF + \sqrt{p_k q_k}X_k) \cdot |-D_kL^{-1}(F - \mathbb{E}[F])|]\\ &= \sum_{k=1}^{\infty} \frac{1}{\sqrt{p_k q_k}}\,\mathbb{E}[D_kF(D_kF + \sqrt{p_k q_k}(p_k - q_k)) \cdot |-D_kL^{-1}(F - \mathbb{E}[F])|].\end{aligned}$$

Plugging this into (3.1) concludes the proof. □

In the following, we will deduce explicit bounds on the error in the Poisson approximation of suitably shifted discrete multiple stochastic integrals of fixed order with respect to the total variation distance. We start with suitably shifted discrete multiple stochastic integrals of order 1. Again, note that the following Theorem 3.4 and Corollary 3.5 are related to Theorem 7.1 in [15].

Theorem 3.4. *Let* $F = \mathbb{E}[F] + J_1(f)$ *with values in* $\mathbb{N}_0$ *and* $f \in \ell^2(\mathbb{N})$. *Furthermore, let* $\mathrm{Po}(\lambda)$ *be a Poisson random variable with mean* $\lambda > 0$. *Then,*

$$\begin{aligned} &d_{TV}(F, \mathrm{Po}(\lambda)) \\ &\leq \Big(1 \wedge \sqrt{\frac{2}{e\lambda}}\Big)|\lambda - \mathbb{E}[F]| + \frac{1-e^{-\lambda}}{\lambda}|\lambda - \|f\|^2_{\ell^2(\mathbb{N})}| \\ &\quad + \frac{1-e^{-\lambda}}{\lambda}\sum_{k=1}^{\infty}\frac{1}{\sqrt{p_kq_k}}(|f^3(k)| + \sqrt{p_kq_k}(p_k - q_k)f^2(k)) \qquad (3.6) \\ &= \Big(1 \wedge \sqrt{\frac{2}{e\lambda}}\Big)|\lambda - \mathbb{E}[F]| + \frac{1-e^{-\lambda}}{\lambda}|\lambda - \mathrm{Var}(F)| \\ &\quad + \frac{1-e^{-\lambda}}{\lambda}\sum_{k=1}^{\infty}\frac{1}{\sqrt{p_kq_k}}(f^2(k) + \sqrt{p_kq_k}(p_k - q_k)f(k)) \cdot |f(k)|. \qquad (3.7) \end{aligned}$$

Proof. In order to show (3.6) and (3.7), we have to evaluate the last two summands of the bound in (3.1). By virtue of Corollary 3.3, we thus have to compute the quantities

$$A_1 := \mathbb{E}[|\lambda - \langle DF, -DL^{-1}(F - \mathbb{E}[F])\rangle_{\ell^2(\mathbb{N})}|]$$

and

$$A_2 := \sum_{k=1}^{\infty}\frac{1}{\sqrt{p_kq_k}}\mathbb{E}[D_kF(D_kF + \sqrt{p_kq_k}(p_k - q_k)) \cdot |-D_kL^{-1}(F - \mathbb{E}[F])|].$$

Now, for every $k \in \mathbb{N}$, we have that

$$D_kF = -D_kL^{-1}(F - \mathbb{E}[F]) = f(k).$$

This yields

$$\langle DF, -DL^{-1}(F - \mathbb{E}[F])\rangle_{\ell^2(\mathbb{N})} = \sum_{k=1}^{\infty} f^2(k) = \|f\|^2_{\ell^2(\mathbb{N})}.$$

In addition, by the isometry formula in (2.6), it follows that

$$\mathrm{Var}(F) = \|f\|^2_{\ell^2(\mathbb{N})}.$$

Thus,

$$A_1 = |\lambda - \|f\|^2_{\ell^2(\mathbb{N})}| = |\lambda - \mathrm{Var}(F)|.$$

Furthermore, we have

$$A_2 = \sum_{k=1}^{\infty} \frac{1}{\sqrt{p_k q_k}} (f^2(k) + \sqrt{p_k q_k}(p_k - q_k) f(k)) \cdot |f(k)|.$$

This concludes the proof. □

Now, the following corollary is a first application of Theorem 3.4 and serves as an insight into the quality of our main bound in Theorem 3.1.

Corollary 3.5. *Let $(B_k)_{k\in\mathbb{N}}$ be a sequence of independent Bernoulli random variables with $P(B_k = 1) = p_k$ and $P(B_k = 0) = q_k$, for every $k \in \mathbb{N}$, and $\sum_{k=1}^{\infty} p_k < \infty$. Furthermore, let $F = \sum_{k=1}^{\infty} B_k$ and let $\mathrm{Po}(\lambda)$ be a Poisson random variable with mean $\lambda > 0$. Then,*

$$d_{TV}(F, \mathrm{Po}(\lambda))$$
$$\leq \Big(1 \wedge \sqrt{\frac{2}{e\lambda}}\Big)\Big|\lambda - \sum_{k=1}^{\infty} p_k\Big| + \frac{1-e^{-\lambda}}{\lambda}\Big|\lambda - \sum_{k=1}^{\infty} p_k q_k\Big| + \frac{2(1-e^{-\lambda})}{\lambda} \sum_{k=1}^{\infty} p_k^2 q_k.$$

Proof. Since, for every $k \in \mathbb{N}$,

$$B_k \stackrel{d}{=} \frac{X_k + 1}{2},$$

F has a representation of the form $F \stackrel{d}{=} \mathbb{E}[F] + J_1(f)$ with $f \in \ell^2(\mathbb{N})$. More precisely, we have

$$F \stackrel{d}{=} \sum_{k=1}^{\infty} \frac{X_k + 1}{2} = \sum_{k=1}^{\infty} p_k + \sum_{k=1}^{\infty} \sqrt{p_k q_k} \frac{X_k + 1 - 2p_k}{2\sqrt{p_k q_k}} = \mathbb{E}[F] + J_1(f)$$

with $f(k) := \sqrt{p_k q_k}$, for every $k \in \mathbb{N}$. Note that $f \in \ell^2(\mathbb{N})$, since

$$\sum_{k=1}^{\infty} f^2(k) = \sum_{k=1}^{\infty} p_k q_k \leq \sum_{k=1}^{\infty} p_k < \infty.$$

According to Theorem 3.4, we thus have to evaluate the quantities

$$A_1 := |\lambda - \mathbb{E}[F]|, \quad A_2 := |\lambda - \mathrm{Var}(F)|$$

and

$$A_3 := \sum_{k=1}^{\infty} \frac{1}{\sqrt{p_k q_k}} (f^2(k) + \sqrt{p_k q_k}(p_k - q_k) f(k)) \cdot |f(k)|.$$

Now,

$$\mathbb{E}[F] = \sum_{k=1}^{\infty} p_k, \quad \mathrm{Var}(F) = \sum_{k=1}^{\infty} p_k q_k.$$

Thus,

$$A_1 = \Big|\lambda - \sum_{k=1}^{\infty} p_k\Big|, \quad A_2 = \Big|\lambda - \sum_{k=1}^{\infty} p_k q_k\Big|.$$

In addition,

$$A_3 = \sum_{k=1}^{\infty} p_k q_k (1 + p_k - q_k) = 2 \sum_{k=1}^{\infty} p_k^2 q_k.$$

This concludes the proof. □

Remark 3.6. Note that, for $\lambda := \sum_{k=1}^{\infty} p_k$, the bound in Corollary 3.5 yields

$$d_{TV}(F, \mathrm{Po}(\lambda)) \leq \frac{1 - e^{-\lambda}}{\lambda} \sum_{k=1}^{\infty} p_k^2 + \frac{2(1 - e^{-\lambda})}{\lambda} \sum_{k=1}^{\infty} p_k^2 q_k \leq \frac{3(1 - e^{-\lambda})}{\lambda} \sum_{k=1}^{\infty} p_k^2,$$

and thus, is (up to the constant) of the quality of the classical result

$$d_{TV}\Big(\sum_{k=1}^{n} B_k, \mathrm{Po}(\lambda)\Big) \leq \frac{1 - e^{-\lambda}}{\lambda} \sum_{k=1}^{n} p_k^2$$

as discussed in Chapter 1 in [1]. However, Corollary 7.1 in [15] does lead to a suboptimal result (cf. Chapter 7 in [15]).

We will now turn to suitably shifted discrete stochastic integrals of order $m \geq 2$. Here, we will have to fully make use of the generalized product formula in Proposition 2.2. For a corresponding result on the Poisson approximation of perturbed functionals of general Poisson measures inside a fixed chaos see Theorem 4.10 in [12]. Again, note that the following Theorem 3.7 and Remark 3.8 are related to Theorem 8.2 and Proposition 8.3, respectively, in [15].

Theorem 3.7. *Let $m \geq 2$ be an integer, $F = \mathbb{E}[F] + J_m(f)$ with values in $\mathbb{N}_0$ and $f \in \ell_0^2(\mathbb{N})^{\circ m}$ fulfilling $(\varphi^{*r-\ell} \widetilde{(f \star_r^\ell f)}) \mathbb{1}_{\Delta_{n+m-r-\ell}} \in \ell_0^2(\mathbb{N})^{\circ n+m-r-\ell}$, for every $r = 1, \ldots, n \wedge m$ and $\ell = 0, \ldots, r-1$. Furthermore, let $\mathrm{Po}(\lambda)$ be a Poisson random variable with mean $\lambda > 0$. Then,*

$$\begin{aligned}
&d_{TV}(F, \mathrm{Po}(\lambda)) \\
&\leq \Big(1 \wedge \sqrt{\frac{2}{e\lambda}}\Big) |\lambda - \mathbb{E}[F]| + \frac{1 - e^{-\lambda}}{\lambda} |\lambda - \mathrm{Var}(F)| \\
&\quad + \frac{1 - e^{-\lambda}}{\lambda} \Big(m^2 \sum_{s=1}^{2(m-1)} s! \Big\| \sum_{r=1}^{m} \sum_{\ell=1}^{r} \mathbb{1}_{\{2m-r-\ell=s\}} (r-1)! \binom{m-1}{r-1}^2 \binom{r-1}{\ell-1} \\
&\qquad\qquad \times (\varphi^{*r-\ell} \widetilde{(f \star_r^\ell f)}) \mathbb{1}_{\Delta_{2m-r-\ell}} \Big\|_{\ell^2(\mathbb{N})^{\otimes s}}^2 \Big)^{1/2} \\
&\quad + \frac{1 - e^{-\lambda}}{\lambda} \sqrt{\mathrm{Var}(F)} \Big(m^3 \sum_{k=1}^{\infty} \frac{1}{p_k q_k} ((m-1)! \| f(\,\cdot\,, k) \|_{\ell^2(\mathbb{N})^{\otimes m-1}}^2)^2 \\
&\qquad + m^3 \sum_{k=1}^{\infty} \frac{1}{p_k q_k} \sum_{\substack{s=1 \\ s \neq m-1}}^{2(m-1)} s! \Big\| \sum_{r=1}^{m} \sum_{\ell=1}^{r} \mathbb{1}_{\{2m-r-\ell=s\}} (r-1)! \binom{m-1}{r-1}^2 \binom{r-1}{\ell-1}
\end{aligned}$$

$$\times \left(\varphi^{*r-\ell}(\widetilde{f(\,\cdot\,,k) \star_{r-1}^{\ell-1}} f(\,\cdot\,,k))\right) \mathbb{1}_{\Delta_{2m-r-\ell}} \Big\|^2_{\ell^2(\mathbb{N})^{\otimes s}}$$

$$+ m^3 \sum_{k=1}^{\infty} \frac{1}{p_k q_k} (m-1)! \Big\| \sum_{r=1}^{m} \sum_{\ell=1}^{r} \mathbb{1}_{\{2m-r-\ell=m-1\}} (r-1)! \binom{m-1}{r-1}^2 \binom{r-1}{\ell-1}$$

$$\times \left(\varphi^{*r-\ell}(\widetilde{f(\,\cdot\,,k) \star_{r-1}^{\ell-1}} f(\,\cdot\,,k))\right) \mathbb{1}_{\Delta_{2m-r-\ell}}$$

$$+ \frac{1}{m} \sqrt{p_k q_k} (p_k - q_k) f(\,\cdot\,,k) \Big\|^2_{\ell^2(\mathbb{N})^{\otimes m-1}} \Bigg)^{1/2}. \quad (3.8)$$

Proof. It suffices to prove (3.8) for kernels $f \in \ell_0^2(\mathbb{N})^{\circ m}$ with finite support only. The general case then follows by considering the sequence of truncated kernels $(f_k)_{k\in\mathbb{N}}$ with $f_k := f\,\mathbb{1}_{\{1,\dots,k\}^n}$, for every $k \in \mathbb{N}$, and the approximation arguments further discussed in Lemma 4.1 and Corollary 4.3 in the appendix. Again, we make use of Corollary 3.3 and compute the quantities

$$A_2 := \mathbb{E}[|\lambda - \langle DF, -DL^{-1}(F - \mathbb{E}[F])\rangle_{\ell^2(\mathbb{N})}|]$$

and

$$A_3 := \sum_{k=1}^{\infty} \frac{1}{\sqrt{p_k q_k}} \mathbb{E}[D_k F (D_k F + \sqrt{p_k q_k}(p_k - q_k)) \cdot |-D_k L^{-1}(F - \mathbb{E}[F])|].$$

Now, for every $k \in \mathbb{N}$, we have

$$D_k F = m J_{m-1}(f(\,\cdot\,,k)). \quad (3.9)$$

By the product formula in (2.4), it then follows that, for every $k \in \mathbb{N}$,

$$\begin{aligned}
(D_k F)^2 &= m^2 (J_{m-1}(f(\,\cdot\,,k)))^2 \\
&= m^2 \sum_{r=0}^{m-1} r! \binom{m-1}{r}^2 \sum_{\ell=0}^{r} \binom{r}{\ell} \\
&\qquad \times J_{2(m-1)-r-\ell}\Big((\varphi^{*r-\ell}(\widetilde{f(\,\cdot\,,k) \star_r^\ell} f(\,\cdot\,,k))) \mathbb{1}_{\Delta_{2(m-1)-r-\ell}} \Big) \\
&= m^2 \sum_{r=1}^{m} (r-1)! \binom{m-1}{r-1}^2 \sum_{\ell=1}^{r} \binom{r-1}{\ell-1} \\
&\qquad \times J_{2m-r-\ell}\Big((\varphi^{*r-\ell}(\widetilde{f(\,\cdot\,,k) \star_{r-1}^{\ell-1}} f(\,\cdot\,,k))) \mathbb{1}_{\Delta_{2m-r-\ell}} \Big). \quad (3.10)
\end{aligned}$$

Thus,

$$\sum_{k=1}^{\infty} (D_k F)^2 = m^2 \sum_{r=1}^{m} (r-1)! \binom{m-1}{r-1}^2 \sum_{\ell=1}^{r} \binom{r-1}{\ell-1}$$

$$\times J_{2m-r-\ell}\Big((\varphi^{*r-\ell}(\widetilde{f \star_r^\ell} f)) \mathbb{1}_{\Delta_{2m-r-\ell}} \Big)$$

$$= m^2 \sum_{s=0}^{2(m-1)} J_s\Big(\sum_{r=1}^{m}\sum_{\ell=1}^{r} \mathbb{1}_{\{2m-r-\ell=s\}} (r-1)! \binom{m-1}{r-1}^2 \binom{r-1}{\ell-1} \times (\varphi^{*r-\ell} \widetilde{(f \star_r^\ell f)})\, \mathbb{1}_{\Delta_{2m-r-\ell}} \Big)$$

$$= m\cdot m! \|f\|^2_{\ell^2(\mathbb{N})^{\otimes m}} + m^2 \sum_{s=1}^{2(m-1)} J_s\Big(\sum_{r=1}^{m}\sum_{\ell=1}^{r} \mathbb{1}_{\{2m-r-\ell=s\}} (r-1)! \binom{m-1}{r-1}^2 \binom{r-1}{\ell-1} \times (\varphi^{*r-\ell} \widetilde{(f \star_r^\ell f)})\, \mathbb{1}_{\Delta_{2m-r-\ell}} \Big). \tag{3.11}$$

Furthermore, for every $k \in \mathbb{N}$, we have

$$-D_k L^{-1}(F - \mathbb{E}[F]) = J_{m-1}(f(\,\cdot\,,k)) = \frac{1}{m} D_k F, \tag{3.12}$$

and therefore, by (3.11)

$$\langle DF, -DL^{-1}(F-\mathbb{E}[F])\rangle_{\ell^2(\mathbb{N})} = \frac{1}{m}\sum_{k=1}^{\infty} (D_k F)^2$$

$$= m!\|f\|^2_{\ell^2(\mathbb{N})^{\otimes m}} + m \sum_{s=1}^{2(m-1)} J_s\Big(\sum_{r=1}^{m}\sum_{\ell=1}^{r} \mathbb{1}_{\{2m-r-\ell=s\}} (r-1)! \binom{m-1}{r-1}^2 \binom{r-1}{\ell-1} \times (\varphi^{*r-\ell} \widetilde{(f \star_r^\ell f)})\, \mathbb{1}_{\Delta_{2m-r-\ell}} \Big).$$

By the Cauchy-Schwarz inequality and the isometry formula in (2.6), we then get

$$\begin{aligned} A_2 &\le (\mathbb{E}[(\lambda - \langle DF, -DL^{-1}(F-\mathbb{E}[F])\rangle_{\ell^2(\mathbb{N})})^2])^{1/2} \\ &\le |\lambda - m!\|f\|^2_{\ell^2(\mathbb{N})^{\otimes m}}| \\ &\quad + \Big(m^2 \sum_{s=1}^{2(m-1)} s! \Big\| \sum_{r=1}^{m}\sum_{\ell=1}^{r} \mathbb{1}_{\{2m-r-\ell=s\}} (r-1)! \binom{m-1}{r-1}^2 \binom{r-1}{\ell-1} \\ &\qquad \times (\varphi^{*r-\ell} \widetilde{(f \star_r^\ell f)})\, \mathbb{1}_{\Delta_{2m-r-\ell}} \Big\|^2_{\ell^2(\mathbb{N})^{\otimes s}} \Big)^{1/2}. \end{aligned}$$

Using (3.12), the Cauchy-Schwarz inequality and (3.11), we further deduce

$$\begin{aligned} A_3 &= \frac{1}{m}\sum_{k=1}^{\infty} \frac{1}{\sqrt{p_k q_k}}\, \mathbb{E}[((D_kF)^2 + \sqrt{p_k q_k}(p_k - q_k) D_k F)\cdot |D_k F|] \\ &\le \frac{1}{m}\Big(\sum_{k=1}^{\infty} \frac{1}{p_k q_k}\, \mathbb{E}[((D_kF)^2 + \sqrt{p_k q_k}(p_k - q_k) D_k F)^2] \Big)^{1/2} \Big(\sum_{k=1}^{\infty} \mathbb{E}[(D_k F)^2] \Big)^{1/2} \end{aligned}$$

$$= (m!\|f\|^2_{\ell^2(\mathbb{N})^{\otimes m}})^{1/2}\Big(\frac{1}{m}\sum_{k=1}^{\infty}\frac{1}{p_kq_k}\,\mathbb{E}[((D_kF)^2+\sqrt{p_kq_k}(p_k-q_k)D_kF)^2]\Big)^{1/2}.$$

Now, by (3.10) and (3.9), we have

$$\begin{aligned}
&(D_kF)^2+\sqrt{p_kq_k}(p_k-q_k)D_kF\\
&= m^2\sum_{r=1}^{m}(r-1)!\binom{m-1}{r-1}^2\sum_{\ell=1}^{r}\binom{r-1}{\ell-1}\\
&\qquad\qquad \times J_{2m-r-\ell}\Big((\varphi^{*r-\ell}(f(\,\cdot\,,k)\,\widetilde{\star_{r-1}^{\ell-1}}\,f(\,\cdot\,,k)))\,\mathbb{1}_{\Delta_{2m-r-\ell}}\Big)\\
&\quad + m\sqrt{p_kq_k}(p_k-q_k)J_{m-1}(f(\,\cdot\,,k))\\
&= m^2\sum_{\substack{s=0\\ s\neq m-1}}^{2(m-1)}J_s\Big(\sum_{r=1}^{m}\sum_{\ell=1}^{r}\mathbb{1}_{\{2m-r-\ell=s\}}(r-1)!\binom{m-1}{r-1}^2\binom{r-1}{\ell-1}\\
&\qquad\qquad\qquad\qquad \times(\varphi^{*r-\ell}(f(\,\cdot\,,k)\,\widetilde{\star_{r-1}^{\ell-1}}\,f(\,\cdot\,,k)))\,\mathbb{1}_{\Delta_{2m-r-\ell}}\Big)\\
&\quad + m^2J_{m-1}\Big(\sum_{r=1}^{m}\sum_{\ell=1}^{r}\mathbb{1}_{\{2m-r-\ell=m-1\}}(r-1)!\binom{m-1}{r-1}^2\binom{r-1}{\ell-1}\\
&\qquad\qquad\qquad\qquad \times(\varphi^{*r-\ell}(f(\,\cdot\,,k)\,\widetilde{\star_{r-1}^{\ell-1}}\,f(\,\cdot\,,k)))\,\mathbb{1}_{\Delta_{2m-r-\ell}}\\
&\qquad\qquad + \frac{1}{m}\sqrt{p_kq_k}(p_k-q_k)f(\,\cdot\,,k)\Big)
\end{aligned}$$

$$\begin{aligned}
&= m^2(m-1)!\|f(\,\cdot\,,k)\|^2_{\ell^2(\mathbb{N})^{\otimes m-1}}\\
&\quad + m^2\sum_{\substack{s=1\\ s\neq m-1}}^{2(m-1)}J_s\Big(\sum_{r=1}^{m}\sum_{\ell=1}^{r}\mathbb{1}_{\{2m-r-\ell=s\}}(r-1)!\binom{m-1}{r-1}^2\binom{r-1}{\ell-1}\\
&\qquad\qquad\qquad\qquad \times(\varphi^{*r-\ell}(f(\,\cdot\,,k)\,\widetilde{\star_{r-1}^{\ell-1}}\,f(\,\cdot\,,k)))\,\mathbb{1}_{\Delta_{2m-r-\ell}}\Big)\\
&\quad + m^2J_{m-1}\Big(\sum_{r=1}^{m}\sum_{\ell=1}^{r}\mathbb{1}_{\{2m-r-\ell=m-1\}}(r-1)!\binom{m-1}{r-1}^2\binom{r-1}{\ell-1}\\
&\qquad\qquad\qquad\qquad \times(\varphi^{*r-\ell}(f(\,\cdot\,,k)\,\widetilde{\star_{r-1}^{\ell-1}}\,f(\,\cdot\,,k)))\,\mathbb{1}_{\Delta_{2m-r-\ell}}\\
&\qquad\qquad + \frac{1}{m}\sqrt{p_kq_k}(p_k-q_k)f(\,\cdot\,,k)\Big).
\end{aligned}$$

Thus, by the isometry formula in (2.6),

$$\begin{aligned}
&\mathbb{E}[((D_kF)^2 + \sqrt{p_kq_k}(p_k - q_k)D_kF)^2] \\
&= m^4((m-1)!\|f(\,\cdot\,,k)\|^2_{\ell^2(\mathbb{N})^{\otimes m-1}})^2 \\
&\quad + m^4 \sum_{\substack{s=1\\ s\neq m-1}}^{2(m-1)} s! \Big\| \sum_{r=1}^{m}\sum_{\ell=1}^{r} \mathbb{1}_{\{2m-r-\ell=s\}}(r-1)!\binom{m-1}{r-1}^2\binom{r-1}{\ell-1} \\
&\qquad\qquad \times (\varphi^{*r-\ell}(\widetilde{f(\,\cdot\,,k)\star^{\ell-1}_{r-1}} f(\,\cdot\,,k)))\,\mathbb{1}_{\Delta_{2m-r-\ell}}\Big\|^2_{\ell^2(\mathbb{N})^{\otimes s}} \\
&\quad + m^4(m-1)!\Big\| \sum_{r=1}^{m}\sum_{\ell=1}^{r} \mathbb{1}_{\{2m-r-\ell=m-1\}}(r-1)!\binom{m-1}{r-1}^2\binom{r-1}{\ell-1} \\
&\qquad\qquad \times (\varphi^{*r-\ell}(\widetilde{f(\,\cdot\,,k)\star^{\ell-1}_{r-1}} f(\,\cdot\,,k)))\,\mathbb{1}_{\Delta_{2m-r-\ell}} \\
&\qquad\qquad + \frac{1}{m}\sqrt{p_kq_k}(p_k-q_k)f(\,\cdot\,,k)\Big\|^2_{\ell^2(\mathbb{N})^{\otimes m-1}},
\end{aligned}$$

and therefore,

$$\begin{aligned}
A_3 &\leq (m!\|f\|^2_{\ell^2(\mathbb{N})^{\otimes m}})^{1/2} \\
&\quad \times \Big(m^3 \sum_{k=1}^{\infty} \frac{1}{p_kq_k}((m-1)!\|f(\,\cdot\,,k)\|^2_{\ell^2(\mathbb{N})^{\otimes m-1}})^2 \\
&\quad + m^3 \sum_{k=1}^{\infty} \frac{1}{p_kq_k} \sum_{\substack{s=1\\ s\neq m-1}}^{2(m-1)} s! \Big\| \sum_{r=1}^{m}\sum_{\ell=1}^{r} \mathbb{1}_{\{2m-r-\ell=s\}}(r-1)!\binom{m-1}{r-1}^2\binom{r-1}{\ell-1} \\
&\qquad\qquad \times (\varphi^{*r-\ell}(\widetilde{f(\,\cdot\,,k)\star^{\ell-1}_{r-1}} f(\,\cdot\,,k)))\,\mathbb{1}_{\Delta_{2m-r-\ell}}\Big\|^2_{\ell^2(\mathbb{N})^{\otimes s}} \\
&\quad + m^3 \sum_{k=1}^{\infty} \frac{1}{p_kq_k}(m-1)!\Big\| \sum_{r=1}^{m}\sum_{\ell=1}^{r} \mathbb{1}_{\{2m-r-\ell=m-1\}}(r-1)!\binom{m-1}{r-1}^2\binom{r-1}{\ell-1} \\
&\qquad\qquad \times (\varphi^{*r-\ell}(\widetilde{f(\,\cdot\,,k)\star^{\ell-1}_{r-1}} f(\,\cdot\,,k)))\,\mathbb{1}_{\Delta_{2m-r-\ell}} \\
&\qquad\qquad + \frac{1}{m}\sqrt{p_kq_k}(p_k-q_k)f(\,\cdot\,,k)\Big\|^2_{\ell^2(\mathbb{N})^{\otimes m-1}}\Big)^{1/2}.
\end{aligned}$$

The result now follows by a final application of the isometry formula in (2.6) to deduce

$$\mathrm{Var}(F) = m!\|f\|^2_{\ell^2(\mathbb{N})^{\otimes m}}.$$

□

Remark 3.8. Resorting, e.g., to the case $m=2$ in Theorem 3.7 yields the bound

$$\begin{aligned}
&d_{TV}(F,\mathrm{Po}(\lambda))\\
&\le \Big(1\wedge\sqrt{\frac{2}{e\lambda}}\Big)|\lambda-\mathbb{E}[F]|+\frac{1-e^{-\lambda}}{\lambda}|\lambda-\mathrm{Var}(F)|\\
&\quad+\frac{1-e^{-\lambda}}{\lambda}(4\|\varphi^{*1}(f\star_2^1 f)\|^2_{\ell^2(\mathbb{N})}+8\|(f\star_1^1 f)\,\mathbb{1}_{\Delta_2}\|^2_{\ell^2(\mathbb{N})^{\otimes 2}})^{1/2}\\
&\quad+\frac{1-e^{-\lambda}}{\lambda}\sqrt{\mathrm{Var}(F)}\\
&\qquad\times\Big(8\sum_{k=1}^{\infty}\frac{1}{p_kq_k}\|f(\,\cdot\,,k)\|^4_{\ell^2(\mathbb{N})}+16\sum_{k=1}^{\infty}\frac{1}{p_kq_k}\|(f(\,\cdot\,,k)\star_0^0 f(\,\cdot\,,k))\,\mathbb{1}_{\Delta_2}\|^2_{\ell^2(\mathbb{N})^{\otimes 2}}\\
&\qquad+8\sum_{k=1}^{\infty}\frac{1}{p_kq_k}\|\varphi^{*1}(f(\,\cdot\,,k)\star_1^0 f(\,\cdot\,,k))+\frac{1}{2}\sqrt{p_kq_k}(p_k-q_k)f(\,\cdot\,,k)\|^2_{\ell^2(\mathbb{N})}\Big)^{1/2}.
\end{aligned}$$

Thus, the weak convergence of the law of $F_n=\mathbb{E}[F_n]+J_2(f_n)$ with $f_n\in\ell_0^2(\mathbb{N})^{\circ 2}$, for every $n\in\mathbb{N}$, to a Poisson distribution is implied by the convergence of the first two moments of F_n and by the vanishing of the quantities

$$\|\varphi^{*1}(f_n\star_2^1 f_n)\|^2_{\ell^2(\mathbb{N})},\quad \|(f_n\star_1^1 f_n)\,\mathbb{1}_{\Delta_2}\|^2_{\ell^2(\mathbb{N})^{\otimes 2}},$$

$$\sum_{k=1}^{\infty}\frac{1}{p_kq_k}\|f_n(\,\cdot\,,k)\|^4_{\ell^2(\mathbb{N})},\quad \sum_{k=1}^{\infty}\frac{1}{p_kq_k}\|(f_n(\,\cdot\,,k)\star_0^0 f_n(\,\cdot\,,k))\,\mathbb{1}_{\Delta_2}\|^2_{\ell^2(\mathbb{N})^{\otimes 2}}$$

and

$$\sum_{k=1}^{\infty}\frac{1}{p_kq_k}\|\varphi^{*1}(f_n(\,\cdot\,,k)\star_1^0 f_n(\,\cdot\,,k))+\frac{1}{2}\sqrt{p_kq_k}(p_k-q_k)f_n(\,\cdot\,,k)\|^2_{\ell^2(\mathbb{N})},$$

as $n\to\infty$.

Corollary 3.9. *Let $n\ge 2$ be an integer, $p_k=1-q_k:=\frac{1}{n}$, for every $k\in\mathbb{N}$, and $F_n:=J_2(f_n)$ with $f_n\in\ell_0^2(\mathbb{N})^{\circ 2}$ given by*

$$f_n(i,j):=\begin{cases}\frac{n-1}{2n^2}, & \text{if } (i,j)\in\{(1,2),(2,1),\dots,(1,n),(n,1)\},\\ 0, & \text{otherwise.}\end{cases}$$

Furthermore, let $\mathrm{Po}(\lambda_n)$ be a Poisson random variable with mean $\lambda_n:=\mathrm{Var}(F_n)$. Then,

$$d_{TV}(F_n,\mathrm{Po}(\lambda_n))\le\frac{C}{\sqrt{n}}$$

with $C:=\frac{5}{2}+\sqrt{2}$.

Proof. First note that F_n fulfills the assumptions of Theorem 3.7. To see that F_n only takes values in $\mathbb{N}_0$, let $(B_k)_{k\in\mathbb{N}}$ be a sequence of independent Bernoulli random variables with $P(B_k = 1) = \frac{1}{n}$ and $P(B_k = 0) = 1 - \frac{1}{n}$, for every $k \in \mathbb{N}$. Then, $Y_k \overset{d}{=} \frac{B_k - p_k}{\sqrt{p_k q_k}} = \frac{nB_k - 1}{\sqrt{n-1}}$, for every $k \in \mathbb{N}$, and thus,

$$F_n = \sum_{i,j=1}^{n} f_n(i,j) Y_i Y_j = \frac{n-1}{n^2} Y_1 \sum_{i=2}^{n} Y_i \overset{d}{=} (B_1 - n) \sum_{i=2}^{n} (B_i - n). \tag{3.13}$$

Now, since $n \geq 2$, we have that $B_i - n$ is a strictly negative integer, for every $i = 1, \ldots, n$. Therefore, it follows from (3.13) that F_n is a strictly positive integer. Let us come to the proof of the assertion. According to Remark 3.8, we have to further compute the quantities

$$A_1(n) := |\lambda_n - \mathbb{E}[F_n]|, \; A_2(n) := |\lambda_n - \mathrm{Var}(F_n)|, \; A_3(n) := \|\varphi^{*1}(f_n \star_2^1 f_n)\|^2_{\ell^2(\mathbb{N})},$$

$$A_4(n) := \|(f_n \star_1^1 f_n) \mathbb{1}_{\Delta_2}\|^2_{\ell^2(\mathbb{N})^{\otimes 2}}, \quad A_5(n) := \sum_{k=1}^{\infty} \frac{1}{p_k q_k} \|f_n(\,\cdot\,, k)\|^4_{\ell^2(\mathbb{N})},$$

$$A_6(n) := \sum_{k=1}^{\infty} \frac{1}{p_k q_k} \|(f_n(\,\cdot\,, k) \star_0^0 f_n(\,\cdot\,, k)) \mathbb{1}_{\Delta_2}\|^2_{\ell^2(\mathbb{N})^{\otimes 2}}$$

and

$$A_7(n) := \sum_{k=1}^{\infty} \frac{1}{p_k q_k} \|\varphi^{*1}(f_n(\,\cdot\,, k) \star_1^0 f_n(\,\cdot\,, k)) + \frac{1}{2}\sqrt{p_k q_k}(p_k - q_k) f_n(\,\cdot\,, k)\|^2_{\ell^2(\mathbb{N})}.$$

First of, since $\lambda_n = \mathrm{Var}(F_n) = 2\|f_n\|^2_{\ell^2(\mathbb{N})^{\otimes 2}} = \frac{(n-1)^3}{n^4}$, we have that $A_1(n) = \frac{(n-1)^3}{n^4} \leq \frac{1}{n}$ and $A_2(n) = 0$. Considering $A_3(n)$, for every $i \in \mathbb{N}$, we get

$$\begin{aligned} f_n \star_2^1 f_n(i) &= \sum_{j=1}^{n} f_n^2(i,j) = f_n^2(i,1) + \sum_{j=2}^{n} f_n^2(i,j) \\ &= \frac{(n-1)^2}{4n^4} \Big(\mathbb{1}_{\{i=2,\ldots,n\}} + (n-1) \mathbb{1}_{\{i=1\}} \Big), \end{aligned}$$

and hence, with $\varphi_k^2 = \frac{(q_k - p_k)^2}{p_k q_k} = \frac{(n-2)^2}{n-1}$, for every $k \in \mathbb{N}$,

$$\begin{aligned} A_3(n) &= \frac{(n-1)^3 (n-2)^2}{16 n^8} \sum_{i=1}^{n} \Big(\mathbb{1}_{\{i=2,\ldots,n\}} + (n-1)^2 \mathbb{1}_{\{i=1\}} \Big) \\ &= \frac{(n-1)^4 (n-2)^2}{16 n^7} \leq \frac{1}{16n}. \end{aligned}$$

Turning to $A_4(n)$, for every $i, j \in \mathbb{N}$, we have

$$\begin{aligned} &(f_n \star_1^1 f_n) \mathbb{1}_{\Delta_2}(i,j) = \sum_{k=1}^{n} f_n(i,k) f_n(j,k) \mathbb{1}_{\Delta_2}(i,j) \\ &= f_n(i,1) f_n(j,1) \mathbb{1}_{\Delta_2}(i,j) + \sum_{k=2}^{n} f_n(i,k) f_n(j,k) \mathbb{1}_{\Delta_2}(i,j) = \frac{(n-1)^2}{4n^4} \mathbb{1}_{\{2 \leq i \neq j \leq n\}}, \end{aligned}$$

and thus,

$$A_4(n) = \frac{(n-1)^5(n-2)}{16n^8} \leq \frac{1}{16n^2}.$$

To compute $A_5(n)$, note that, for every $k \in \mathbb{N}$,

$$\begin{aligned}\|f_n(\,\cdot\,,k)\|^4_{\ell^2(\mathbb{N})} &= \Big(f_n^2(1,k) + \sum_{j=2}^n f_n^2(j,k)\Big)^2 \\ &= \frac{(n-1)^4}{16n^8}\Big(\mathbb{1}_{\{k=2,\dots,n\}} + (n-1)^2\,\mathbb{1}_{\{k=1\}}\Big),\end{aligned}$$

and therefore, with $\frac{1}{p_k q_k} = \frac{n^2}{n-1}$, for every $k \in \mathbb{N}$,

$$A_5(n) = \frac{(n-1)^3}{16n^6}\sum_{k=1}^n \Big(\mathbb{1}_{\{k=2,\dots,n\}} + (n-1)^2\,\mathbb{1}_{\{k=1\}}\Big) = \frac{(n-1)^4}{16n^5} \leq \frac{1}{16n}.$$

For $A_6(n)$, it shows that, for every $i,j,k \in \mathbb{N}$,

$$\begin{aligned}(f_n(\,\cdot\,,k) \star_0^0 f_n(\,\cdot\,,k))\,\mathbb{1}_{\Delta_2}(i,j) &= f_n(i,k)f_n(j,k)\,\mathbb{1}_{\Delta_2}(i,j) \\ &= \frac{(n-1)^2}{4n^4}\,\mathbb{1}_{\{2\leq i\neq j\leq n\}}\,\mathbb{1}_{\{k=1\}}.\end{aligned}$$

Furthermore, for every $k \in \mathbb{N}$,

$$\|(f_n(\,\cdot\,,k) \star_0^0 f_n(\,\cdot\,,k))\,\mathbb{1}_{\Delta_2}\|^2_{\ell^2(\mathbb{N})^{\otimes 2}} = \frac{(n-1)^5(n-2)}{16n^8}\,\mathbb{1}_{\{k=1\}},$$

so that, again with $\frac{1}{p_k q_k} = \frac{n^2}{n-1}$, for every $k \in \mathbb{N}$,

$$A_6(n) = \frac{(n-1)^4(n-2)}{16n^6} \leq \frac{1}{16n}.$$

Finally, considering $A_7(n)$, for every $k \in \mathbb{N}$, it holds that

$$\begin{aligned}&\|\varphi^{*1}(f_n(\,\cdot\,,k) \star_1^0 f_n(\,\cdot\,,k)) + \frac{1}{2}\sqrt{p_k q_k}(p_k - q_k)f_n(\,\cdot\,,k)\|^2_{\ell^2(\mathbb{N})} \\ &= \sum_{j=1}^n \Big(\frac{n-2}{\sqrt{n-1}}f_n^2(j,k) - \frac{\sqrt{n-1}(n-2)}{2n^2}f_n(j,k)\Big)^2 \\ &= \Big(\frac{(n-1)^{3/2}(n-2)}{4n^4} - \frac{(n-1)^{3/2}(n-2)}{4n^4}\Big)^2\Big(\mathbb{1}_{\{k=2,\dots,n\}} + (n-1)\,\mathbb{1}_{\{k=1\}}\Big) = 0,\end{aligned}$$

where we used that $\varphi_k = \frac{n-2}{\sqrt{n-1}}$ and $\sqrt{p_k q_k}(p_k - q_k) = -\frac{\sqrt{n-1}(n-2)}{n^2}$, for every $k \in \mathbb{N}$. We thus conclude that $A_7(n) = 0$. Now, it follows from Remark 3.8 that

$$d_{TV}(F_n, \mathrm{Po}(\lambda_n)) \leq A_1(n) + 2\sqrt{A_3(n)} + 2\sqrt{2A_4(n)} + 2\sqrt{2A_5(n)} + 4\sqrt{A_6(n)},$$

where we used that, for every $n \geq 2$, $\Big(1 \wedge \sqrt{\frac{2}{e\lambda_n}}\Big) = 1$, $\frac{1-e^{-\lambda_n}}{\lambda_n} \leq 1$ and $\sqrt{\mathrm{Var}(F_n)} \leq 1$. This yields the assertion. $\square$

Remark 3.10. Note here that the Rademacher functional in Corollary 3.9 is of the same spirit as the one considered in the example that follows Proposition 8.3 in [15]. For a sequence of success probabilities $p = (p_k)_{k\in\mathbb{N}}$ as in Corollary 3.9, the authors of [15] compare a suitably shifted stochastic double integral $F_n = \lambda_n + J_2(f_n)$ to a Poisson random variable $\mathrm{Po}(\lambda_n)$, where $\lambda_n \geq 4n$ is an integer and $f_n \in \ell_0^2(\mathbb{N})^{\circ 2}$ is given by

$$f_n(i,j) := \begin{cases} \frac{n-1}{n}, & \text{if } (i,j) \in \{(1,2),(2,1),\ldots,(1,n),(n,1)\}, \\ 0, & \text{otherwise.} \end{cases}$$

However, for this particular choice our bound in Remark 3.8 as well as the corresponding bound in Proposition 8.3 in [15] does not vanish, as $n \to \infty$, since the involved norms of contractions do not tend to zero. For example, we have

$$2\|f_n\|^2_{\ell^2(\mathbb{N})^{\otimes 2}} = \frac{4(n-1)^3}{n^2} \quad \text{and} \quad \|f_n \star_1^1 f_n\|^2_{\ell^2(\mathbb{N})^{\otimes 2}} = \frac{2(n-1)^6}{n^4},$$

so that the quantities $\frac{1-e^{-\lambda_n}}{\lambda_n}|\lambda_n - 2\|f_n\|^2_{\ell^2(\mathbb{N})^{\otimes 2}}|$ and $\frac{1-e^{-\lambda_n}}{\lambda_n}\|f_n \star_1^1 f_n\|^2_{\ell^2(\mathbb{N})^{\otimes 2}}$ in the bound of Proposition 8.3 in [15] never vanish at the same time, no matter of the choice of λ_n.

We will now turn to our final result, a second order Poincaré type bound for the Poisson approximation of Rademacher functionals. One advantage of such a bound is that it can be further evaluated without the use of a product formula for multiple stochastic integrals or even a specification of the chaos representation of the Rademacher functional of interest as in (2.2). See, e.g., Theorem 1.1 in [7] for an efficient application of a corresponding second order Poincaré type bound for the normal approximation of Rademacher functionals. Before we come to the statement, we collect some tools from [7].

Lemma 3.11 (cf. Proposition 3.3 in [7])**.** *For $m \in \mathbb{N}$, let $k_1,\ldots,k_m \in \mathbb{N}$ and $F \in \mathrm{dom}(D^m)$. Then, for every real $\alpha \geq 1$,*

$$\mathbb{E}[|D^m_{k_1,\ldots,k_m} L^{-1}(F - \mathbb{E}[F])|^\alpha] \leq \mathbb{E}[|D^m_{k_1,\ldots,k_m} F|^\alpha].$$

Lemma 3.12 (cf. Proposition 3.4 and Remark 3.1 in [7])**.** *Let $F \in L^1(\Omega)$. Then,*

$$\mathrm{Var}(F) \leq \mathbb{E}[\|DF\|^2_{\ell^2(\mathbb{N})}].$$

Theorem 3.13. *Let $F \in \mathrm{dom}(D^2)$ with values in $\mathbb{N}_0$ and let $\mathrm{Po}(\lambda)$ be a Poisson random variable with mean $\lambda > 0$. Then,*

$$\begin{aligned} &d_{TV}(F, \mathrm{Po}(\lambda)) \\ &\leq \Big(1 \wedge \sqrt{\frac{2}{e\lambda}}\Big)|\lambda - \mathbb{E}[F]| + \frac{1-e^{-\lambda}}{\lambda}|\lambda - \mathrm{Var}(F)| \\ &\quad + \frac{1-e^{-\lambda}}{\lambda}\Big(\frac{15}{4}\sum_{j,k,\ell=1}^{\infty} (\mathbb{E}[(D_jF)^2(D_kF)^2])^{1/2}(\mathbb{E}[(D_\ell D_jF)^2(D_\ell D_kF)^2])^{1/2}\Big)^{1/2} \\ &\quad + \frac{1-e^{-\lambda}}{\lambda}\Big(\frac{3}{4}\sum_{j,k,\ell=1}^{\infty} \frac{1}{p_\ell q_\ell}\,\mathbb{E}[(D_\ell D_jF)^2(D_\ell D_kF)^2]\Big)^{1/2} \end{aligned}$$

$$+\frac{1-e^{-\lambda}}{\lambda}\sum_{k=1}^{\infty}\frac{1}{\sqrt{p_kq_k}}(\mathbb{E}[(D_kF)^2(D_kF+\sqrt{p_kq_k}(p_k-q_k))^2])^{1/2}(\mathbb{E}[(D_kF)^2])^{1/2}. \tag{3.14}$$

Proof. We build on Corollary 3.3 by further estimating the quantities

$$A_1 := \mathbb{E}[|\lambda - \langle DF, -DL^{-1}(F-\mathbb{E}[F])\rangle_{\ell^2(\mathbb{N})}|]$$

and

$$A_2 := \sum_{k=1}^{\infty}\frac{1}{\sqrt{p_kq_k}}\,\mathbb{E}[D_kF(D_kF+\sqrt{p_kq_k}(p_k-q_k))\cdot|-D_kL^{-1}(F-\mathbb{E}[F])|].$$

Starting with A_1, by means of the triangle and the Cauchy-Schwarz inequality, we get

$$\begin{aligned} A_1 &\le \mathbb{E}[|\lambda-\mathrm{Var}(F)|]+\mathbb{E}[|\mathrm{Var}(F)-\langle DF, DL^{-1}(F-\mathbb{E}[F])\rangle_{\ell^2(\mathbb{N})}|] \\ &\le \mathbb{E}[|\lambda-\mathrm{Var}(F)|]+(\mathbb{E}[(\mathrm{Var}(F)-\langle DF, DL^{-1}(F-\mathbb{E}[F])\rangle_{\ell^2(\mathbb{N})})^2])^{1/2}. \end{aligned} \tag{3.15}$$

Note that, by choosing $G = F-\mathbb{E}[F]$ in the integration by parts formula in (2.12), we have

$$\mathrm{Var}(F) = \mathbb{E}[\langle DF, DL^{-1}(F-\mathbb{E}[F])\rangle_{\ell^2(\mathbb{N})})^2],$$

and thus,

$$\mathbb{E}[(\mathrm{Var}(F)-\langle DF, DL^{-1}(F-\mathbb{E}[F])\rangle_{\ell^2(\mathbb{N})})^2] = \mathrm{Var}(\langle DF, DL^{-1}(F-\mathbb{E}[F])\rangle_{\ell^2(\mathbb{N})}).$$

Hence, the second summand on the right hand side of (3.15) can be further estimated by Lemma 3.12 and Lemma 3.11 as shown in the proof of Theorem 4.1 in [7], which leads to

$$\begin{aligned} A_1 &\le \mathbb{E}[|\lambda-\mathrm{Var}(F)|] \\ &\quad + \Big(\frac{15}{4}\sum_{j,k,\ell=1}^{\infty}(\mathbb{E}[(D_jF)^2(D_kF)^2])^{1/2}(\mathbb{E}[(D_\ell D_jF)^2(D_\ell D_kF)^2])^{1/2}\Big)^{1/2} \\ &\quad + \Big(\frac{3}{4}\sum_{j,k,\ell=1}^{\infty}\frac{1}{p_\ell q_\ell}\,\mathbb{E}[(D_\ell D_jF)^2(D_\ell D_kF)^2]\Big)^{1/2}. \end{aligned}$$

Furthermore, by virtue of the Cauchy-Schwarz inequality and Lemma 3.11, we get

$$\begin{aligned} A_2 &\le \sum_{k=1}^{\infty}\frac{1}{\sqrt{p_kq_k}}(\mathbb{E}[(D_kF)^2(D_kF+\sqrt{p_kq_k}(p_k-q_k))^2])^{1/2} \\ &\qquad\qquad \times(\mathbb{E}[(D_kL^{-1}(F-\mathbb{E}[F]))^2])^{1/2} \\ &\le \sum_{k=1}^{\infty}\frac{1}{\sqrt{p_kq_k}}(\mathbb{E}[(D_kF)^2(D_kF+\sqrt{p_kq_k}(p_k-q_k))^2])^{1/2}(\mathbb{E}[(D_kF)^2])^{1/2}. \end{aligned}$$

This concludes the proof. □

Remark 3.14. To give a first application and an insight into the quality of the bound in Theorem 3.13, we consider the Poisson approximation of infinite sums of Bernoulli random variables once more. For this, let $(B_k)_{k\in\mathbb{N}}$ be a sequence of independent Bernoulli random variables with $P(B_k = 1) = p_k$ and $P(B_k = 0) = q_k$, for every $k \in \mathbb{N}$, and $\sum_{k=1}^{\infty} p_k < \infty$, and let $F := \sum_{k=1}^{\infty} B_k$. Recall from the proof of Corollary 3.5 that

$$F \overset{d}{=} \sum_{k=1}^{\infty} \frac{X_k + 1}{2}.$$

Now, for every $k \in \mathbb{N}$, we have that

$$F_k^+ \overset{d}{=} 1 + \sum_{\substack{\ell=1\\ \ell\neq k}}^{\infty} \frac{X_\ell + 1}{2} \quad \text{and} \quad F_k^- \overset{d}{=} \sum_{\substack{\ell=1\\ \ell\neq k}}^{\infty} \frac{X_\ell + 1}{2},$$

and therefore, for every $k, \ell \in \mathbb{N}$, we get

$$D_k F = \sqrt{p_k q_k}(F_k^+ - F_k^-) = \sqrt{p_k q_k} \quad \text{and} \quad D_\ell D_k F = 0,$$

P-almost surely. Hence, $F \in \text{dom}(D^2)$ by (2.8) and all assumptions of Theorem 3.13 are fulfilled. According to this, we have to further compute the quantities

$$A_1 := |\lambda - \mathbb{E}[F]|, \quad A_2 := |\lambda - \text{Var}(F)|,$$

$$A_3 := \Big(\sum_{j,k,\ell=1}^{\infty} (\mathbb{E}[(D_j F)^2 (D_k F)^2])^{1/2} (\mathbb{E}[(D_\ell D_j F)^2 (D_\ell D_k F)^2])^{1/2} \Big)^{1/2},$$

$$A_4 := \Big(\sum_{j,k,\ell=1}^{\infty} \frac{1}{p_\ell q_\ell} \, \mathbb{E}[(D_\ell D_j F)^2 (D_\ell D_k F)^2] \Big)^{1/2}$$

and

$$A_5 := \sum_{k=1}^{\infty} \frac{1}{\sqrt{p_k q_k}} (\mathbb{E}[(D_k F)^2 (D_k F + \sqrt{p_k q_k}(p_k - q_k))^2])^{1/2} (\mathbb{E}[(D_k F)^2])^{1/2}$$

from the bound in (3.14). Recall from the proof of Corollary 3.5 that

$$A_1 = \Big|\lambda - \sum_{k=1}^{\infty} p_k\Big| \quad \text{and} \quad A_2 = \Big|\lambda - \sum_{k=1}^{\infty} p_k q_k\Big|.$$

Furthermore, $A_3 = A_4 = 0$ and

$$A_5 = \sum_{k=1}^{\infty} p_k q_k (1 + p_k - q_k) = 2 \sum_{k=1}^{\infty} p_k^2 q_k.$$

This leads to the exact same bound

$$d_{TV}(F, \text{Po}(\lambda)) \le \Big(1 \wedge \sqrt{\frac{2}{e\lambda}}\Big) \Big|\lambda - \sum_{k=1}^{\infty} p_k\Big| + \frac{1 - e^{-\lambda}}{\lambda} \Big|\lambda - \sum_{k=1}^{\infty} p_k q_k\Big| + \frac{2(1 - e^{-\lambda})}{\lambda} \sum_{k=1}^{\infty} p_k^2 q_k$$

that we have deduced directly from Theorem 3.1 in Corollary 3.5.

4. Appendix

The purpose of this appendix is to prove Proposition 2.2. We start by collecting some arguments that will be used within the proof. Note that Lemma 4.1 is a slight generalization of Lemma 2.6 in [11], while Lemma 4.2 is known as Lemma 4.6 in [14].

Lemma 4.1. *Fix $n, m \in \mathbb{N}$. Furthermore, let $(f_k)_{k\in\mathbb{N}}$ and $(g_k)_{k\in\mathbb{N}}$ be two sequences of kernels with $f_k \in \ell_0^2(\mathbb{N})^{\circ n}$ and $g_k \in \ell_0^2(\mathbb{N})^{\circ m}$, for every $k \in \mathbb{N}$. Then, if $(f_k)_{k\in\mathbb{N}}$ converges to a kernel f in $\ell_0^2(\mathbb{N})^{\circ n}$ and $(g_k)_{k\in\mathbb{N}}$ converges to a kernel g in $\ell_0^2(\mathbb{N})^{\circ m}$, it holds that, for every $r = 0, \dots, n \wedge m$ and $\ell = 0, \dots, r$, the sequence of contractions $(f_k \star_r^\ell g_k)_{k\in\mathbb{N}}$ converges to $f \star_r^\ell g$ in $\ell^2(\mathbb{N})^{\otimes n+m-r-\ell}$.*

Proof. Using the triangle inequality as well as Lemma 2.4 in [11], we see that

$$\begin{aligned}
&\|f_k \star_r^\ell g_k - f \star_r^\ell g\|_{\ell^2(\mathbb{N})^{\otimes n+m-r-\ell}} \\
&= \|f_k \star_r^\ell (g_k - g + g) - (f - f_k + f_k) \star_r^\ell g\|_{\ell^2(\mathbb{N})^{\otimes n+m-r-\ell}} \\
&= \|f_k \star_r^\ell (g_k - g) + (f_k - f) \star_r^\ell g\|_{\ell^2(\mathbb{N})^{\otimes n+m-r-\ell}} \\
&\le \|f_k \star_r^\ell (g_k - g)\|_{\ell^2(\mathbb{N})^{\otimes m+n-r-\ell}} + \|(f_k - f) \star_r^\ell g\|_{\ell^2(\mathbb{N})^{\otimes m+n-r-\ell}} \\
&\le \|f_k\|_{\ell^2(\mathbb{N})^{\otimes n}} \|g_k - g\|_{\ell^2(\mathbb{N})^{\otimes m}} + \|f_k - f\|_{\ell^2(\mathbb{N})^{\otimes n}} \|g\|_{\ell^2(\mathbb{N})^{\otimes m}}.
\end{aligned}$$

The statement now follows immediately by taking the limit $k \to \infty$. □

Lemma 4.2. *Let $n \in \mathbb{N}$ and $f \in \ell_0^2(\mathbb{N})^{\circ n}$. Consider the sequence of truncated kernels $(f_k)_{k\in\mathbb{N}}$ with $f_k := f\, \mathbb{1}_{\{1,\dots,k\}^n}$, for every $k \in \mathbb{N}$. Then, for every $k \in \mathbb{N}$,*

$$J_n(f_k) = \mathbb{E}[J_n(f)|\mathcal{F}_k],$$

where $(\mathcal{F}_k)_{k\in\mathbb{N}}$ denotes the canonical filtration given by $\mathcal{F}_k := \sigma(X_1, \dots, X_k)$, for every $k \in \mathbb{N}$.

Corollary 4.3. *Let $n \in \mathbb{N}$ and $f \in \ell_0^2(\mathbb{N})^{\circ n}$. Consider the sequence of truncated kernels $(f_k)_{k\in\mathbb{N}}$ with $f_k := f\, \mathbb{1}_{\{1,\dots,k\}^n}$, for every $k \in \mathbb{N}$. Then, the sequence $(J_n(f_k))_{k\in\mathbb{N}}$ convergences to $J_n(f)$ in $L^2(\Omega)$.*

Proof. By virtue of Lemma 4.2, $(J_n(f_k))_{k\in\mathbb{N}}$ is a martingale with respect to the filtration $(\mathcal{F}_k)_{k\in\mathbb{N}}$. Thus, the convergence of $(J_n(f_k))_{k\in\mathbb{N}}$ to $J_n(f)$ in $L^2(\Omega)$ immediately follows by the martingale convergence theorem. □

Proof of Proposition 2.2. Fix $d \in \mathbb{N}$. We start by proving (2.4) for stochastic integrals of kernels $f \in \ell_0^2(\mathbb{N})^{\circ n}$ and $g \in \ell_0^2(\mathbb{N})^{\circ m}$ with finite supports $\text{supp}(f) \subseteq \{1, \dots, d\}^n$ and $\text{supp}(g) \subseteq \{1, \dots, d\}^m$. We put $\Delta_n^d := \Delta_n \cap \{1, \dots, d\}^n$ and deduce from (2.1) that

$$\begin{aligned}
&J_n(f)J_m(g) \\
&= \sum_{(i_1,\dots,i_n,j_1,\dots,j_m)\in\Delta_n^d\times\Delta_m^d} f(i_1, \dots, i_n)g(j_1, \dots, j_m)Y_{i_1} \cdot \ldots \cdot Y_{i_n} Y_{j_1} \cdot \ldots \cdot Y_{j_m}.
\end{aligned} \tag{4.1}$$

We will now count the pairs of equal random variables in the products $Y_{i_1} \cdot \ldots \cdot Y_{i_n} Y_{j_1} \cdot \ldots \cdot Y_{j_m}$ in (4.1). Since $(i_1, \ldots, i_n) \in \Delta_n^d$ and $(j_1, \ldots, j_m) \in \Delta_m^d$, each possible pair can only consist of one random variable taken from the set $\{Y_{i_1}, \ldots, Y_{i_n}\}$ and one random variable taken from the set $\{Y_{j_1}, \ldots, Y_{j_m}\}$. Thus, each product $Y_{i_1} \cdot \ldots \cdot Y_{i_n} Y_{j_1} \cdot \ldots \cdot Y_{j_m}$ can contain $r = 0, \ldots, n \wedge m$ pairs. Now, there are $r!\binom{n}{r}\binom{m}{r}$ different ways to build r pairs as described above. (There are $\binom{n}{r}$ different ways to pick r random variables from $\{Y_{i_1}, \ldots, Y_{i_n}\}$, $\binom{m}{r}$ different ways to pick r random variables from $\{Y_{j_1}, \ldots, Y_{j_m}\}$ and finally $r!$ different ways to group pairs from the two developed r-sets.) By the symmetry of the summands $f(i_1, \ldots, i_n)g(j_1, \ldots, j_m)Y_{i_1} \cdot \ldots \cdot Y_{i_n} Y_{j_1} \cdot \ldots \cdot Y_{j_m}$ in $i_1, \ldots, i_n$ and $j_1, \ldots, j_m$, respectively, the sum in (4.1) can be rewritten in terms of summands containing r pairs of random variables

$$J_n(f)J_m(g) = \sum_{r=0}^{n\wedge m} r!\binom{n}{r}\binom{m}{r} \sum_{(\boldsymbol{i}_{n-r}, \boldsymbol{j}_{m-r}, \boldsymbol{k}_r) \in \Delta_{n+m-r}^d} f(\boldsymbol{i}_{n-r}, \boldsymbol{k}_r) g(\boldsymbol{j}_{m-r}, \boldsymbol{k}_r) \times Y_{i_1} \cdot \ldots \cdot Y_{i_{n-r}} Y_{j_1} \cdot \ldots \cdot Y_{j_{m-r}} Y_{k_1}^2 \cdot \ldots \cdot Y_{k_r}^2 \tag{4.2}$$

with $\boldsymbol{i}_{n-r} := (i_1, \ldots, i_{n-r})$, $\boldsymbol{j}_{m-r} := (j_1, \ldots, j_{m-r})$ and $\boldsymbol{k}_r := (k_1, \ldots, k_r)$. We will now further compute the product $Y_{k_1}^2 \cdot \ldots \cdot Y_{k_r}^2$ in (4.2). By (2.3) it follows that

$$\prod_{\ell=1}^r Y_{k_\ell}^2 = \prod_{\ell=1}^r (1 + \varphi_{k_\ell} Y_{k_\ell}) = 1 + \sum_{s=1}^r \sum_{1 \le \ell_1 < \ldots < \ell_s \le r} \varphi_{k_{\ell_1}} \cdot \ldots \cdot \varphi_{k_{\ell_s}} Y_{k_{\ell_1}} \cdot \ldots \cdot Y_{k_{\ell_s}}.$$

Thus, the inner sum in (4.2) can be rewritten as the sum of the two quantities

$$\sum_{(\boldsymbol{i}_{n-r}, \boldsymbol{j}_{m-r}, \boldsymbol{k}_r) \in \Delta_{n+m-r}^d} f(\boldsymbol{i}_{n-r}, \boldsymbol{k}_r) g(\boldsymbol{j}_{m-r}, \boldsymbol{k}_r) Y_{i_1} \cdot \ldots \cdot Y_{i_{n-r}} Y_{j_1} \cdot \ldots \cdot Y_{j_{m-r}} \tag{4.3}$$

and

$$\sum_{(\boldsymbol{i}_{n-r}, \boldsymbol{j}_{m-r}, \boldsymbol{k}_r) \in \Delta_{n+m-r}^d} \sum_{s=1}^r \sum_{1 \le \ell_1 < \ldots < \ell_s \le r} \varphi_{k_{\ell_1}} \cdot \ldots \cdot \varphi_{k_{\ell_s}} f(\boldsymbol{i}_{n-r}, \boldsymbol{k}_r) g(\boldsymbol{j}_{m-r}, \boldsymbol{k}_r) \times Y_{i_1} \cdot \ldots \cdot Y_{i_{n-r}} Y_{j_1} \cdot \ldots \cdot Y_{j_{m-r}} Y_{k_{\ell_1}} \cdot \ldots \cdot Y_{k_{\ell_s}}. \tag{4.4}$$

Using the fact that f and g vanish on diagonals as well as the symmetry of the product measure $\mu_{(Y,d)}^{\otimes n+m-2r}$ defined by $\mu_{(Y,d)}^{\otimes n+m-2r}(A) := \sum_{(i_1, \ldots, i_{n+m-2r}) \in A} Y_{i_1} \cdot \ldots \cdot Y_{i_{n+m-2r}}$, for every $A \in \{1, \ldots, d\}^{n+m-2r}$, (4.3) can be further deduced as

$$\sum_{(\boldsymbol{i}_{n-r}, \boldsymbol{j}_{m-r}, \boldsymbol{k}_r) \in \Delta_{n+m-r}^d} f(\boldsymbol{i}_{n-r}, \boldsymbol{k}_r) g(\boldsymbol{j}_{m-r}, \boldsymbol{k}_r) Y_{i_1} \cdot \ldots \cdot Y_{i_{n-r}} Y_{j_1} \cdot \ldots \cdot Y_{j_{m-r}}$$
$$= \sum_{(\boldsymbol{i}_{n-r}, \boldsymbol{j}_{m-r}) \in \Delta_{n+m-2r}^d} \sum_{\boldsymbol{k}_r \in \Delta_r^d} f(\boldsymbol{i}_{n-r}, \boldsymbol{k}_r) g(\boldsymbol{j}_{m-r}, \boldsymbol{k}_r) Y_{i_1} \cdot \ldots \cdot Y_{i_{n-r}} Y_{j_1} \cdot \ldots \cdot Y_{j_{m-r}}$$

$$= \sum_{(\boldsymbol{i}_{n-r},\boldsymbol{j}_{m-r})\in\Delta^d_{n+m-2r}} f \star_r^r g(\boldsymbol{i}_{n-r},\boldsymbol{j}_{m-r}) Y_{i_1}\cdot\ldots\cdot Y_{i_{n-r}} Y_{j_1}\cdot\ldots\cdot Y_{j_{m-r}}$$

$$= \sum_{(\boldsymbol{i}_{n-r},\boldsymbol{j}_{m-r})\in\Delta^d_{n+m-2r}} (\widetilde{f \star_r^r g})(\boldsymbol{i}_{n-r},\boldsymbol{j}_{m-r}) Y_{i_1}\cdot\ldots\cdot Y_{i_{n-r}} Y_{j_1}\cdot\ldots\cdot Y_{j_{m-r}}$$

$$= J_{n+m-2r}\left((\widetilde{f \star_r^r g})\,\mathbb{1}_{\Delta^d_{n+m-2r}}\right). \tag{4.5}$$

To further compute (4.4), note that, due to the symmetry of f and g, the summands

$$\varphi_{k_{\ell_1}}\cdot\ldots\cdot\varphi_{k_{\ell_s}} f(\boldsymbol{i}_{n-r},\boldsymbol{k}_r) g(\boldsymbol{j}_{m-r},\boldsymbol{k}_r) Y_{i_1}\cdot\ldots\cdot Y_{i_{n-r}} Y_{j_1}\cdot\ldots\cdot Y_{j_{m-r}} Y_{k_{\ell_1}}\cdot\ldots\cdot Y_{k_{\ell_s}}$$

are symmetric in $k_1,\ldots,k_r$. Thus, we get that, for $r = 1,\ldots,n\wedge m$,

$$\begin{aligned}
&\sum_{\boldsymbol{k}_r\in\Delta^d_r}\sum_{1\le\ell_1<\ldots<\ell_s\le r} \varphi_{k_{\ell_1}}\cdot\ldots\cdot\varphi_{k_{\ell_s}} f(\boldsymbol{i}_{n-r},\boldsymbol{k}_r) g(\boldsymbol{j}_{m-r},\boldsymbol{k}_r)\\
&\qquad\qquad \times Y_{i_1}\cdot\ldots\cdot Y_{i_{n-r}} Y_{j_1}\cdot\ldots\cdot Y_{j_{m-r}} Y_{k_{\ell_1}}\cdot\ldots\cdot Y_{k_{\ell_s}}\\
&= \binom{r}{s}\sum_{\boldsymbol{k}_s\in\Delta^d_s} \varphi_{k_1}\cdot\ldots\cdot\varphi_{k_s} f \star_r^{r-s} g(\boldsymbol{i}_{n-r},\boldsymbol{k}_s,\boldsymbol{j}_{m-r})\\
&\qquad\qquad \times Y_{i_1}\cdot\ldots\cdot Y_{i_{n-r}} Y_{j_1}\cdot\ldots\cdot Y_{j_{m-r}} Y_{k_1}\cdot\ldots\cdot Y_{k_s}\\
&= \binom{r}{s}\sum_{\boldsymbol{k}_s\in\Delta^d_s} \varphi^{*s}(f \star_r^{r-s} g)(\boldsymbol{i}_{n-r},\boldsymbol{k}_s,\boldsymbol{j}_{m-r})\\
&\qquad\qquad \times Y_{i_1}\cdot\ldots\cdot Y_{i_{n-r}} Y_{j_1}\cdot\ldots\cdot Y_{j_{m-r}} Y_{k_1}\cdot\ldots\cdot Y_{k_s}.
\end{aligned}$$

Therefore, by using the same arguments as in (4.5), we obtain for (4.4) that, if $r = 1,\ldots n\wedge m$,

$$\begin{aligned}
&\sum_{(\boldsymbol{i}_{n-r},\boldsymbol{j}_{m-r},\boldsymbol{k}_r)\in\Delta^d_{n+m-r}}\sum_{s=1}^{r}\sum_{1\le\ell_1<\ldots<\ell_s\le r} \varphi_{k_{\ell_1}}\cdot\ldots\cdot\varphi_{k_{\ell_s}} f(\boldsymbol{i}_{n-r},\boldsymbol{k}_r) g(\boldsymbol{j}_{m-r},\boldsymbol{k}_r)\\
&\qquad\qquad \times Y_{i_1}\cdot\ldots\cdot Y_{i_{n-r}} Y_{j_1}\cdot\ldots\cdot Y_{j_{m-r}} Y_{k_{\ell_1}}\cdot\ldots\cdot Y_{k_{\ell_s}}\\
&= \sum_{s=1}^{r}\binom{r}{s}\sum_{(\boldsymbol{i}_{n-r},\boldsymbol{j}_{m-r},\boldsymbol{k}_s)\in\Delta^d_{n+m-2r+s}} \varphi^{*s}(f \star_r^{r-s} g)(\boldsymbol{i}_{n-r},\boldsymbol{k}_s,\boldsymbol{j}_{m-r})\\
&\qquad\qquad \times Y_{i_1}\cdot\ldots\cdot Y_{i_{n-r}} Y_{j_1}\cdot\ldots\cdot Y_{j_{m-r}} Y_{k_1}\cdot\ldots\cdot Y_{k_s}\\
&= \sum_{s=1}^{r}\binom{r}{s}\sum_{(\boldsymbol{i}_{n-r},\boldsymbol{j}_{m-r},\boldsymbol{k}_s)\in\Delta^d_{n+m-2r+s}} (\varphi^{*s}\widetilde{(f \star_r^{r-s} g)})(\boldsymbol{i}_{n-r},\boldsymbol{k}_s,\boldsymbol{j}_{m-r})
\end{aligned}$$

$$\times Y_{i_1} \cdot \ldots \cdot Y_{i_{n-r}} Y_{j_1} \cdot \ldots \cdot Y_{j_{m-r}} Y_{k_1} \cdot \ldots \cdot Y_{k_s}$$

$$= \sum_{s=1}^{r} \binom{r}{s} J_{n+m-2r+s}\Big((\varphi^{*s} \widetilde{(f \star_r^{r-s} g)})\, \mathbb{1}_{\Delta^d_{n+m-2r+s}} \Big), \tag{4.6}$$

and, if $r = 0$,

$$\sum_{(\boldsymbol{i}_{n-r}, \boldsymbol{j}_{m-r}, \boldsymbol{k}_r) \in \Delta^d_{n+m-r}} \sum_{s=1}^{r} \sum_{1 \le \ell_1 < \ldots < \ell_s \le r} \varphi_{k_{\ell_1}} \cdot \ldots \cdot \varphi_{k_{\ell_s}} f(\boldsymbol{i}_{n-r}, \boldsymbol{k}_r) g(\boldsymbol{j}_{m-r}, \boldsymbol{k}_r) = 0. \tag{4.7}$$

Plugging (4.5), (4.6) and (4.7) into (4.2) finally yields

$$\begin{aligned}
&J_n(f) J_m(g) \\
&= \sum_{r=0}^{n \wedge m} r! \binom{n}{r} \binom{m}{r} J_{n+m-2r}\left(\widetilde{(f \star_r^r g)}\, \mathbb{1}_{\Delta^d_{n+m-2r}} \right) \\
&\quad + \sum_{r=1}^{n \wedge m} r! \binom{n}{r} \binom{m}{r} \sum_{s=1}^{r} \binom{r}{s} J_{n+m-2r+s}\Big((\varphi^{*s} \widetilde{(f \star_r^{r-s} g)})\, \mathbb{1}_{\Delta^d_{n+m-2r+s}} \Big) \\
&= \sum_{r=0}^{n \wedge m} r! \binom{n}{r} \binom{m}{r} J_{n+m-2r}\left(\widetilde{(f \star_r^r g)}\, \mathbb{1}_{\Delta^d_{n+m-2r}} \right) \\
&\quad + \sum_{r=1}^{n \wedge m} r! \binom{n}{r} \binom{m}{r} \sum_{\ell=0}^{r-1} \binom{r}{\ell} J_{n+m-r-\ell}\Big((\varphi^{*r-\ell} \widetilde{(f \star_r^{\ell} g)})\, \mathbb{1}_{\Delta^d_{n+m-r-\ell}} \Big) \\
&= \sum_{r=0}^{n \wedge m} r! \binom{n}{r} \binom{m}{r} \sum_{\ell=0}^{r} \binom{r}{\ell} J_{n+m-r-\ell}\Big((\varphi^{*r-\ell} \widetilde{(f \star_r^{\ell} g)})\, \mathbb{1}_{\Delta^d_{n+m-r-\ell}} \Big)
\end{aligned}$$

for stochastic integrals of kernels f and g with finite supports $\mathrm{supp}(f) \subseteq \{1, \ldots, d\}^n$ and and $\mathrm{supp}(g) \subseteq \{1, \ldots, d\}^m$. For the general case consider the sequences of truncated kernels $(f_d)_{d \in \mathbb{N}}$ and $(g_d)_{d \in \mathbb{N}}$ with $f_d := f\, \mathbb{1}_{\{1,\ldots,d\}^n}$ and $g_d := g\, \mathbb{1}_{\{1,\ldots,d\}^m}$, for every $d \in \mathbb{N}$. Now, $f_d \in \ell_0^2(\mathbb{N})^{\circ n}$ with $\mathrm{supp}(f_d) \subseteq \{1, \ldots, d\}^n$ and $g_d \in \ell_0^2(\mathbb{N})^{\circ m}$ with $\mathrm{supp}(g_d) \subseteq \{1, \ldots, d\}^m$, for every $d \in \mathbb{N}$. According to Lemma 4.1 and Corollary 4.3, the statement now follows from the discussion above by taking the limit $d \to \infty$. □

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