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A CLARK-OCONE TYPE FORMULA UNDER CHANGE OF MEASURE FOR MULTIDIMENSIONAL LÉVY PROCESSES

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Abstract. In this paper, we derive a Clark-Ocone type formula under change of measure for multidimensional Lévy processes. This is a multidimensional version of [14, 15, 9]. By using it, we obtain explicit representations of locally risk-minimizing hedging strategy for markets driven by multidimensional Lévy processes. This is a generalization of [3].

1. Introduction

The representations of functionals of Lévy processes by stochastic integrals are important theorems in Probability theory. In particular, the Clark-Ocone (in short, CO) formula is an explicit stochastic integral representation for random variables in terms of Malliavin derivatives:

$$
\begin{aligned}
F &= \mathbb{E}[F] + \sum_{j=1}^{d} \int_{[0,T]\times\mathbb{R}} \mathbb{E}[D^j_{t,z}F|\mathcal{F}_{t-}]Q_j(dt,dz) \\
&= \mathbb{E}[F] + \sum_{j=1}^{d} \sigma_j \int_0^T \mathbb{E}[D^j_{t,0}F|\mathcal{F}_{t-}]dW_{j,t} + \sum_{j=1}^{d} \int_0^T \int_{\mathbb{R}_0} \mathbb{E}[D^j_{t,z}F|\mathcal{F}_{t-}]z\tilde{N}_j(dt,dz).
\end{aligned}
$$

We precisely define notations and give sufficient conditions for this formula in section 4. There are many results of CO formulas (see introduction of [9, 14, 15] and [6]). Girsanov transformations versions of CO formulas were also studied by many people because many applications in mathematical finance require representation of random variables with respect to risk neutral martingale measure. In this paper, we derive a Clark-Ocone type formula under change of measure (in short, COCM) for multidimensional Lévy processes:

$$
\begin{aligned}
F = \mathbb{E}_{\mathbb{P}^*}[F] &+ \sum_{j=1}^{d} \sigma_j \int_0^T \mathbb{E}_{\mathbb{P}^*}\left[D^j_{t,0}F - FK^j_t \middle| \mathcal{F}_{t-} \right] dW^{\mathbb{P}^*}_{j,t} \\
&+ \sum_{j=1}^{d} \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[F(H^{j,*}_{t,z} - 1) + zH^{j,*}_{t,z}D^j_{t,z}F|\mathcal{F}_{t-}]\tilde{N}^{\mathbb{P}^*}_j(dt,dz), a.s.
\end{aligned}
$$

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We precisely define notations and give sufficient conditions for this formula in section 5.

On the other hand, locally risk-minimizing hedging strategy (LRM, for short) is a very well-known hedging method for contingent claims in a quadratic way (see [11, 12]). In this paper, we obtain an explicit representation of LRM in an incomplete financial market driven by a multidimensional Lévy process by using Malliavin calculus because in real markets, investors sell an option and want to replicate its payoff $F(T, S_T)$ by trading many stocks (liquid assets). This result is a multidimensional version of Arai-Suzuki [3].

This paper is organized as follows: In Section 2-4, we develop Malliavin calculus for multidimensional Lévy processes. In Section 5, by using results of Section 2-4, we derive a COCM for multidimensional Lévy processes. In Section 6, we consider LRM for multidimensional Lévy markets. In Section 7, by using the result of Section 5, we derive explicit representations of LRM for markets driven by multidimensional Lévy processes.

2. Malliavin Calculus for Multidimensional Canonical Lévy Processes

2.1. Setting. We begin with preparation of the probabilistic framework. Let $T > 0$ be a finite time horizon, $(\Omega_W, \mathcal{F}_W, \mathbb{P}_W)$ a one-dimensional Wiener space on $[0, T]$; and W a one-dimensional standard Brownian motion with $W_0 = 0$. Let $(\Omega_J, \mathcal{F}_J, \mathbb{P}_J)$ be the canonical Lévy space (see Solé et al. [13], Delong and Imkeller [5] and Di Nunno et al. [6]) for a pure jump Lévy process J on $[0, T]$ with Lévy measure ν, that is, $\Omega_J = \cup_{n=0}^{\infty}([0, T] \times \mathbb{R}_0)^n$, where $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$; and $J_t(\omega_J) = \sum_{i=1}^n z_i \mathbf{1}_{\{t_i \le t\}}$ for $t \in [0, T]$ and $\omega_J = ((t_1, z_1), \dots, (t_n, z_n)) \in ([0, T] \times \mathbb{R}_0)^n$. Note that $([0, T] \times \mathbb{R}_0)^0$ represents an empty sequence. Now, we assume that $\int_{\mathbb{R}_0} z^2 \nu(dz) < \infty$; and denote $(\Omega^0, \mathcal{F}^0, \mathbb{P}^0) = (\Omega_W \times \Omega_J, \mathcal{F}_W \times \mathcal{F}_J, \mathbb{P}_W \times \mathbb{P}_J)$ and we call it canonical space. Let $\mathbb{F} = \{\mathcal{F}_t^0\}_{t \in [0,T]}$ be the canonical filtration completed for $\mathbb{P}$. Let X^0 be a square integrable centered Lévy process on $(\Omega^0, \mathcal{F}^0, \mathbb{P}^0)$ represented as

$$X_t^0 = \sigma W_t + J_t - t \int_{\mathbb{R}_0} z \nu(dz), \tag{2.1}$$

where $\sigma > 0$. Denoting by N the Poisson random measure defined as $N(t, A) := \sum_{s \le t} \mathbf{1}_A(\Delta X_s)$, $A \in \mathcal{B}(\mathbb{R}_0)$ and $t \in [0, T]$, where $\Delta X_s := X_s - X_{s-}$, we have $J_t = \int_0^t \int_{\mathbb{R}_0} z N(ds, dz)$. In addition, we define its compensated measure as $\tilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt$. Thus, we can rewrite (2.1) as

$$X_t^0 = \sigma W_t + \int_0^t \int_{\mathbb{R}_0} z \tilde{N}(ds, dz). \tag{2.2}$$

Now, let $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1), \cdots, (\Omega^d, \mathcal{F}^d, \mathbb{P}^d)$ be d independent copies of $(\Omega^0, \mathcal{F}^0, \mathbb{P}^0)$ for some $d \ge 1$. We set $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega_1 \times \cdots \times \Omega_d, \mathcal{F}_1 \times \cdots \times \mathcal{F}_d, \mathbb{P}_1 \times \cdots \times \mathbb{P}_d)$ and we call it multidimensional canonical space. Let $X = (X^1, \cdots, X^d)$ be a d-dimensional square integrable centered Lévy process on $(\Omega, \mathcal{F}, \mathbb{P})$ where $X_t^j = \sigma_j W_{j,t} + \int_0^t \int_{\mathbb{R}_0} z \tilde{N}_j(ds, dz), 1 \le j \le d$ where $\sigma_j > 0$, $W_{j,t}$ a Brownian motion on $(\Omega^j, \mathcal{F}^j, \mathbb{P}^j)$, $\tilde{N}_j$ the compensated Poisson random measure on $(\Omega^j, \mathcal{F}^j, \mathbb{P}^j)$ has Lévy measure ν_j satisfies $\int_{\mathbb{R}_0} z^2 \nu_j(dz) < \infty$.

We consider the finite measure q^j defined on $[0,T] \times \mathbb{R}$ by

$$q^j(E) = \sigma_j^2 \int_{E(0)} dt\delta_0(dz) + \int_{E'} z^2 dt\nu_j(dz), \qquad E \in \mathcal{B}([0,T]\times\mathbb{R}),$$

where $E(0) = \{(t,0) \in [0,T]\times\mathbb{R}; (t,0) \in E\}$ and $E' = E - E(0)$, and the random measure Q_j on $[0,T]\times\mathbb{R}$ by

$$Q_j(E) = \sigma_j \int_{E(0)} dW_{j,t}\delta_0(dz) + \int_{E'} z\tilde{N}_j(dt,dz), \qquad E \in \mathcal{B}([0,T]\times\mathbb{R}).$$

We consider the product of the form $\mathbb{H}_\alpha(\omega) := \prod_{j=1}^d I_{\alpha^{(j)}}(f_{j,\alpha^{(j)}})(\omega_j)$ for any $\alpha \in \mathcal{J}^d$, which is the set of indexes of the form $\alpha = (\alpha^{(1)}, \cdots, \alpha^{(d)})$ with $\alpha^{(j)} = 0, 1, \cdots,$ for $j = 1, \cdots, d$. Here $I_{\alpha^{(j)}}(f_{j,\alpha^{(j)}})$ is the $\alpha^{(j)}$-fold iterated Itô integral with respect to random measure Q:

$$\begin{aligned} &I_{\alpha^{(j)}}(f_{j,\alpha^{(j)}}) \\ &:= \int_{([0,T]\times\mathbb{R})^{\alpha^{(j)}}} f_{j,\alpha^{(j)}}((t_1,z_1), \cdots, (t_{\alpha^{(j)}}, z_{\alpha^{(j)}}))Q_j(dt_1,dz_1)\cdots Q_j(dt_{\alpha^{(j)}}, dz_{\alpha^{(j)}}) \end{aligned}$$

where $f_{j,\alpha^{(j)}}$ is deterministic function satisfying

$$\begin{aligned} &\int_{([0,T]\times\mathbb{R})^{\alpha^{(j)}}} |f_{j,\alpha^{(j)}}((t_1,z_1), \cdots, (t_{\alpha^{(j)}}, z_{\alpha^{(j)}}))|^2 q^j(dt_1,dz_1)\cdots q^j(dt_{\alpha^{(j)}}, dz_{\alpha^{(j)}}) \\ &< \infty. \end{aligned}$$

The elements $\mathbb{H}_\alpha, \alpha \in \mathcal{J}^d$, constitute an orthogonal basis in $L^2(\mathbb{P})$. Any real $\mathcal{F}_T$ -measurable random variable $F \in L^2(\mathbb{P})$ can be written as $F = \sum_{\alpha\in\mathcal{J}^d} \mathbb{H}_\alpha$ for an appropriate choice of deterministic symmetric integrands in the iterated Itô integrals.

Definition 2.1. (1) Let $\mathbb{D}^{1,2}$ denote the set of $\mathcal{F}$ -measurable random variables $F \in L^2(\mathbb{P})$ with the representation

$$F = \sum_{\alpha\in\mathcal{J}^d} \mathbb{H}_\alpha, \mathbb{H}_\alpha = \prod_{j=1}^d I_{\alpha^{(j)}}(f_{j,\alpha^{(j)}})(\omega_j)$$

satisfying

$$\sum_{j=1}^d \sum_{\alpha\in\mathcal{J}^d} \alpha^{(j)}\alpha^{(j)}! \|f_{j,\alpha^{(j)}}\|^2_{L^2(([0,T]\times\mathbb{R})^{\alpha^{(j)}})} < \infty.$$

(2) Let $F \in \mathbb{D}^{1,2}$. Then we define the *Malliavin derivative* DF of a random variable $F \in \mathbb{D}^{1,2}$ as the gradient

$$D_{t,z}F = (D^1_{t,z}F, \cdots, D^d_{t,z}F)$$

where

$$D^j_{t,z}F := \sum_{\alpha\in\mathcal{J}^d} \alpha^{(j)}\mathbb{H}_{\alpha-\epsilon^{(j)}}(t,z), t \in [0,T], z \in \mathbb{R}, j = 1, \cdots, d.$$

Here $\epsilon^{(j)} = (0, \cdots, 0, 1, 0, \cdots, 0)$ with 1 in the jth position.
(3) Let $\mathbb{D}^{j,1,2}, j = 1, \cdots, d$ denote the set of $\mathcal{F}$ -measurable random variables $F \in L^2(\mathbb{P})$ with the representation

$$F = \sum_{\alpha \in \mathcal{J}^d} \mathbb{H}_\alpha, \mathbb{H}_\alpha = \prod_{j=1}^d I_{\alpha^{(j)}}(f_{j,\alpha^{(j)}})(\omega_j)$$

satisfying

$$\sum_{\alpha \in \mathcal{J}^d} \alpha^{(j)} \alpha^{(j)}! \|f_{j,\alpha^{(j)}}\|^2_{L^2(([0,T]\times\mathbb{R})^{\alpha^{(j)}})} < \infty$$

for $j = 1, \cdots, d$.

We next establish the following fundamental result.

Proposition 2.2 (The closability of operator D)**.** *Let $F \in L^2(\mathbb{P})$ and $F_k \in \mathbb{D}^{j,1,2}, k \in \mathbb{N}$ such that*
(1) $\lim_{k\to\infty} F_k = F$ *in* $L^2(\mathbb{P})$,
(2) $\{D^j_{t,z}F_k\}_{k=1}^\infty$ *converges in* $L^2(q^j \times \mathbb{P})$.
Then $F \in \mathbb{D}^{j,1,2}$ *and* $\lim_{k\to\infty} D^j_{t,z}F_k = D^j_{t,z}F$ *in* $L^2(q \times \mathbb{P})$.

Proof. We can show this proposition by the same sort argument as Theorem 12.6 of Di Nunno et al. [6]. Let $F = \sum_{\alpha\in\mathcal{J}^d} \mathbb{H}_\alpha, \mathbb{H}_\alpha = \prod_{j=1}^d I_{\alpha^{(j)}}(f_{j,\alpha^{(j)}})$ and $F^k = \sum_{\alpha\in\mathcal{J}^d} \mathbb{H}^k_\alpha, \mathbb{H}^k_\alpha = \prod_{j=1}^d I_{\alpha^{(j)}}(f^k_{j,\alpha^{(j)}})$. Then by assumption (1), we have

$$\sum_{\alpha \in \mathcal{J}^d} \alpha^{(j)}! \|f_{j,\alpha^{(j)}} - f^k_{j,\alpha^{(j)}}\|^2_{L^2(([0,T]\times\mathbb{R})^{\alpha^{(j)}})} = 0.$$

This implies that $\lim_{k\to\infty} f^k_{j,\alpha^{(j)}} = f_{j,\alpha^{(j)}}$ in $L^2_{T,q,n}$ for all $\alpha \in \mathcal{J}^d$. From assumption (2), we deduce that

$$\begin{aligned}
&\lim_{k,m\to\infty} \sum_{\alpha \in \mathcal{J}^d} \alpha^{(j)} \alpha^{(j)}! \|f^k_{j,\alpha^{(j)}} - f^m_{j,\alpha^{(j)}}\|^2_{L^2(([0,T]\times\mathbb{R})^{\alpha^{(j)}})} \\
&= \lim_{k,m\to\infty} \mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} \left(D^j_{t,z}F_k - D^j_{t,z}F_m\right)^2 q^j(dt,dz)\right] = 0.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&\lim_{k\to\infty} \sum_{\alpha \in \mathcal{J}^d} \alpha^{(j)} \alpha^{(j)}! \|f^k_{j,\alpha^{(j)}} - f_{j,\alpha^{(j)}}\|^2_{L^2(([0,T]\times\mathbb{R})^{\alpha^{(j)}})} \\
&\leq 2 \lim_{k\to\infty} \sum_{\alpha \in \mathcal{J}^d} \alpha^{(j)} \alpha^{(j)}! \liminf_{m\to\infty} \|f^k_{j,\alpha^{(j)}} - f^m_{j,\alpha^{(j)}}\|^2_{L^2(([0,T]\times\mathbb{R})^{\alpha^{(j)}})} \\
&\leq 2 \lim_{k\to\infty} \liminf_{m\to\infty} \sum_{\alpha \in \mathcal{J}^d} \alpha^{(j)} \alpha^{(j)}! \|f^k_{j,\alpha^{(j)}} - f^m_{j,\alpha^{(j)}}\|^2_{L^2(([0,T]\times\mathbb{R})^{\alpha^{(j)}})} = 0.
\end{aligned}$$

Therefore, we can see that $F \in \mathbb{D}^{j,1,2}$ and $\lim_{k\to\infty} D^j_{t,z}F_k = D^j_{t,z}F$ in $L^2(q^j \times \mathbb{P})$. □

We next introduce a chain rule for the Malliavin derivatives.

Proposition 2.3. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}, n \geq 1$ be a C^1-function with bounded derivative. If $F = (F_1, \cdots, F_n) \in (\mathbb{D}^{j,1,2})^n$, then $\varphi(F) \in \mathbb{D}^{j,1,2}$ and*

$$D^j_{t,z}\varphi(F) = \sum_{k=1}^n \frac{\partial \varphi}{\partial x_k}(F) D^j_{t,0}F_k \mathbf{1}_{\{0\}}(z) + \frac{\varphi(F_1 + zD^j_{t,z}F_1, \cdots, F_n + zD^j_{t,z}F_n) - \varphi(F_1, \cdots, F_n)}{z} \mathbf{1}_{\mathbb{R}_0}(z).$$

holds.

Proof. We can show this proposition by the same sort argument as Proposition 2.6 in [14]. □

Proposition 2.4 (Chain rule). *Let $\varphi \in C^1(\mathbb{R}^n; \mathbb{R})$ and $F = (F_1, \cdots, F_n)$, where $F_1, \cdots, F_n \in \mathbb{D}^{j,1,2}$. Suppose that $\varphi(F) \in L^2(\mathbb{P})$ and*

$$\sum_{k=1}^n \frac{\partial}{\partial x_k} \varphi(F) D^j_{t,0}F_k \mathbf{1}_{\{0\}}(z) + \frac{\varphi(F_1 + zD^j_{t,z}F_1, \cdots, F_n + zD^j_{t,z}F_n) - \varphi(F_1, \cdots, F_n)}{z} \mathbf{1}_{\mathbb{R}_0}(z) \in L^2(q^j \times \mathbb{P}).$$

Then we obtain $\varphi(F) \in \mathbb{D}^{j,1,2}$ and

$$D^j_{t,z}\varphi(F) = \sum_{k=1}^n \frac{\partial \varphi}{\partial x_k}(F) D^j_{t,0}F_k \mathbf{1}_{\{0\}}(z) + \frac{\varphi(F_1 + zD^j_{t,z}F_1, \cdots, F_n + zD^j_{t,z}F_n) - \varphi(F_1, \cdots, F_n)}{z} \mathbf{1}_{\mathbb{R}_0}(z).$$

Proof. We can show this proposition by the same sort argument as Lemma A.1 of Ocone-Karatzas [7]. □

If we take $\varphi(x, y) = xy$, then we can derive the following product rule.

Corollary 2.5. *Let $F_1, F_2 \in \mathbb{D}^{j,1,2}$ and $F_1F_2 \in L^2(\mathbb{P})$. Moreover, assume that $F_1D^j_{t,z}F_2 + F_2D^j_{t,z}F_1 + zD^j_{t,z}F_1 \cdot D^j_{t,z}F_2 \in L^2(q^j \times \mathbb{P})$. Then $F_1F_2 \in \mathbb{D}^{1,2}$ and*

$$D^j_{t,z}F_1F_2 = F_1D^j_{t,z}F_2 + F_2D^j_{t,z}F_1 + zD^j_{t,z}F_1 \cdot D^j_{t,z}F_2, \tag{2.3}$$

q^j-a.e. $(t, z) \in [0, T] \times \mathbb{R}, \mathbb{P} - a.s.$

3. Commutation of Integration and the Malliavin Differentiability

In this section, we consider commutation of integration and the Malliavin differentiability.

Definition 3.1. For $1 \leq i, j \leq d$, we define the following: (1) Let $\mathbb{L}^{j,1,2}$ denote the space of product measurable and $\mathbb{F}$ -adapted processes $G_i : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ satisfying

$$\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} |G_{i,s,x}|^2 q^i(ds, dx)\right] < \infty,$$

$G_{i,s,x} \in \mathbb{D}^{j,1,2}, q^i-$a.e. $(s,x) \in [0,T] \times \mathbb{R}$ and

$$\mathbb{E}\left[\int_{([0,T]\times\mathbb{R})^2} |D^j_{t,z} G_{i,s,x}|^2 q^i(ds,dx) q^j(dt,dz)\right] < \infty.$$

(2) $\mathbb{L}^{j,1,2}_0$ denotes the space of $G : [0,T] \times \Omega \to \mathbb{R}$ satisfying
(i) $G_{i,s} \in \mathbb{D}^{j,1,2}$ for a.e. $s \in [0,T]$,
(ii) $E\left[\int_{[0,T]} |G_{i,s}|^2 ds\right] < \infty$,
(iii) $E\left[\int_{[0,T]\times\mathbb{R}} \int_0^T |D^j_{t,z} G_{i,s}|^2 ds q^j(dt,dz)\right] < \infty$.
(3) $\mathbb{L}^{j,1,2}_1$ is defined as the space of $G : [0,T] \times \mathbb{R}_0 \times \Omega \to \mathbb{R}$ such that
(i) $G_{i,s,x} \in \mathbb{D}^{j,1,2}$ for q^i-a.e. $(s,x) \in [0,T] \times \mathbb{R}$,
(ii) $E\left[\int_{[0,T]\times\mathbb{R}_0} |G_{i,s,x}|^2 \nu_i(dx) ds\right] < \infty$,
(iii) $E\left[\int_{[0,T]\times\mathbb{R}} \int_{[0,T]\times\mathbb{R}_0} |D^j_{t,z} G_{i,s,x}|^2 \nu_i(dx) ds q^j(dt,dz)\right] < \infty$.
(4) $\tilde{\mathbb{L}}^{j,1,2}_1$ is defined as the space of $G \in \mathbb{L}^{1,2}$ such that
(i) $E\left[\left(\int_{[0,T]\times\mathbb{R}_0} |G_{i,s,x}| \nu_i(dx) ds\right)^2\right] < \infty$,
(ii) $E\left[\int_{[0,T]\times\mathbb{R}} \left(\int_{[0,T]\times\mathbb{R}_0} |D^j_{t,z} G_{i,s,x}| \nu_i(dx) ds\right)^2 q^j(dt,dz)\right] < \infty$.

We next discuss the commutation relation of the stochastic integral with the Malliavin derivative. By the same arguments of Lemmas 3.2 and 3.3 of Delong and Imkeller [5], we can derive the following:

Proposition 3.2. *Let $G_i : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ be a predictable process with*

$$\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} |G_{i,s,x}|^2 q^i(ds,dx)\right] < \infty.$$

Then

$$G \in \mathbb{L}^{j,1,2} \quad \text{if and only if} \quad \int_{[0,T]\times\mathbb{R}} G_{i,s,x} Q_i(ds,dx) \in \mathbb{D}^{j,1,2}.$$

Furthermore, if $\int_{[0,T]\times\mathbb{R}} G_{s,x} Q_i(ds,dx) \in \mathbb{D}^{1,2}$, then we have

$$D^j_{t,z} \int_{[0,T]\times\mathbb{R}} G_{s,x} Q(ds,dx) = G^i_{t,z} + \int_{[0,T]\times\mathbb{R}} D^j_{t,z} G_{s,x} Q_i(ds,dx), \quad \mathbb{P}-\text{a.s.},$$

for $i = j$ and

$$D^j_{t,z} \int_{[0,T]\times\mathbb{R}} G_{i,s,x} Q_i(ds,dx) = \int_{[0,T]\times\mathbb{R}} D^j_{t,z} G_{s,x} Q_i(ds,dx), \quad \mathbb{P}-\text{a.s.},$$

for $i \neq j$.

Proposition 3.3. *Assume that $G_i : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ is a product measurable and $\mathbb{F}$ -adapted process, η on $[0,T] \times \mathbb{R}$ a finite measure, so that conditions*

$$\begin{aligned}
&\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} |G_{i,s,x}|^2 \eta(ds,dx)\right] < \infty,\\
&G_{i,s,x} \in \mathbb{D}^{j,1,2}, \quad \text{for } \eta\text{−a.e. } (s,x) \in [0,T]\times\mathbb{R},\\
&\mathbb{E}\left[\int_{([0,T]\times\mathbb{R})^2} |D^j_{t,z} G_{i,s,x}|^2 \eta(ds,dx) q(dt,dz)\right] < \infty
\end{aligned}$$

are satisfied. Then we have

$$\int_{[0,T]\times\mathbb{R}} G_{i,s,x}\eta(ds,dx) \in \mathbb{D}^{j,1,2}$$

and the differentiation rule

$$D_{t,z}\int_{[0,T]\times\mathbb{R}} G_{i,s,x}\eta(ds,dx) = \int_{[0,T]\times\mathbb{R}} D^j_{t,z} G_{s,x}\eta(ds,dx)$$

holds for q^j -a.e. $(t,z) \in [0,T]\times\mathbb{R}, \mathbb{P}$ -a.s.

By using σ-finiteness of ν and Proposition 3.3, we can show the following proposition.

Proposition 3.4. *Let $G \in \tilde{\mathbb{L}}^{1,2}_1$. Then*

$$\int_{[0,T]\times\mathbb{R}_0} G_{s,x}\nu(dx)ds \in \mathbb{D}^{1,2}$$

and the differentiation rule

$$D_{t,z}\int_{[0,T]\times\mathbb{R}_0} G_{s,x}\nu(dx)ds = \int_{[0,T]\times\mathbb{R}_0} D_{t,z}G_{s,x}\nu(dx)ds$$

holds for q -a.e. $(t,z) \in [0,T]\times\mathbb{R}, \mathbb{P}$ -a.s.

Proof. We can show the same step as Proposition 3.5 in [14]. □

4. Clark-Ocone Type Formula for Canonical Multidimensional Lévy Functionals and Girsanov Type Theorem

4.1. Clark-Ocone type formula for canonical multidimensional Lévy functionals. We next present an explicit form of the martingale representation formula by using Malliavin calculus (see e.g., Theorem 12.20 in Di Nunno et al. [6]).

Proposition 4.1. *Let $F \in \mathbb{D}^{1,2}$. Then we have*

$$\begin{aligned}
F &= \mathbb{E}[F] + \sum_{j=1}^d \int_{[0,T]\times\mathbb{R}} \mathbb{E}[D^j_{t,z}F|\mathcal{F}_{t-}]Q_j(dt,dz)\\
&= \mathbb{E}[F] + \sum_{j=1}^d \sigma_j \int_0^T \mathbb{E}[D^j_{t,0}F|\mathcal{F}_{t-}]dW_{j,t} + \sum_{j=1}^d \int_0^T\int_{\mathbb{R}_0} \mathbb{E}[D^j_{t,z}F|\mathcal{F}_{t-}]z\tilde{N}_j(dt,dz).
\end{aligned} \tag{4.1}$$

Proof. The martingale representation theorem (see, e.g. Section 2 of Benth et al. [4]) provides that

$$
\begin{aligned}
F &= \mathbb{E}[F] + \sum_{j=1}^{d} \int_0^T \varphi^{(1)}_{j,s-} dW_{j,s} + \sum_{j=1}^{d} \int_0^T \int_{\mathbb{R}_0} \varphi^{(2)}_{j,s-,x} \tilde{N}_j(ds,dx) \\
&= \mathbb{E}[F] + \sum_{j=1}^{d} \int_0^T \frac{\varphi^{(1)}_{j,s-}}{\sigma_j} \sigma dW_{j,s} + \sum_{j=1}^{d} \int_0^T \int_{\mathbb{R}_0} \frac{\varphi^{(2)}_{j,s-,x}}{x} x \tilde{N}_j(ds,dx) \\
&= \mathbb{E}[F] + \sum_{j=1}^{d} \int_0^T \int_{\mathbb{R}} \left(\frac{\varphi^{(1)}_{j,s-}}{\sigma_j} \mathbf{1}_{\{0\}}(x) + \frac{\varphi^{(2)}_{j,s-,x}}{x} \mathbf{1}_{\mathbb{R}_0}(x) \right) Q_j(ds,dx),
\end{aligned}
$$

where $\varphi^{(1)}_j$ and $\varphi^{(2)}_j/x, x \neq 0$ are $L^2(q^j \times \mathbb{P})$-predictable processes. Since $F \in \mathbb{D}^{1,2}$, Proposition 3.3 implies that

$$
\begin{aligned}
D^j_{t,z} F =& \frac{\varphi^{(1)}_{j,t-}}{\sigma_j} \mathbf{1}_{\{0\}}(z) + \frac{\varphi^{(2)}_{j,t-,z}}{z} \mathbf{1}_{\mathbb{R}_0}(z) \\
&+ \sum_{i=1}^{d} \int_{t-}^T \int_{\mathbb{R}} D^j_{t,z} \left(\frac{\varphi^{(1)}_{i,s-}}{\sigma_i} \mathbf{1}_{\{0\}}(x) + \frac{\varphi^{(2)}_{i,s-,x}}{x} \mathbf{1}_{\mathbb{R}_0}(x) \right) Q_i(ds,dx). \qquad (4.2)
\end{aligned}
$$

Hence we have

$$
\mathbb{E}[D^j_{t,z} F | \mathcal{F}_{t-}] = \frac{\varphi^{(1)}_{j,t-}}{\sigma_j} \mathbf{1}_{\{0\}}(z) + \frac{\varphi^{(2)}_{j,t-,z}}{z} \mathbf{1}_{\mathbb{R}_0}(z).
$$

Therefore, we can see that

$$
\begin{aligned}
\varphi^{(1)}_{j,t-} &= \sigma_j \mathbb{E}[D^j_{t,0} F | \mathcal{F}_{t-}] \\
\varphi^{(2)}_{j,t-,z} &= z \mathbb{E}[D^j_{t,z} F | \mathcal{F}_{t-}].
\end{aligned}
$$

$\square$

4.2. Girsanov theorem for Lévy processes. We recall the Girsanov theorem for Lévy processes (see, e.g., Theorem 2.5 of Øksendal and Sulem [8]).

Theorem 4.2. *Let $\theta(s,x) \in \mathbb{R}^d$ with $\theta_{i,s,x} < 1, s \in [0,T], x \in \mathbb{R}_0$ and $u_s \in \mathbb{R}^d, s \in [0,T]$, be predictable processes such that*

$$
\begin{aligned}
&\sum_{i=1}^{d} \int_0^T \int_{\mathbb{R}_0} \{ |\log(1-\theta_{i,s,x})|^2 + \theta^2_{i,s,x} \} \nu_i(dx) ds < \infty, \text{ a.s.}, \\
&\sum_{i=1}^{d} \int_0^T u^2_{i,s} ds < \infty, \text{ a.s.}
\end{aligned}
$$

Moreover, we denote

$$\begin{aligned}Z_t := \exp\Big(&-\sum_{i=1}^d \int_0^t u_{i,s} dW_{i,s} - \frac{1}{2}\sum_{i=1}^d \int_0^t u_{i,s}^2 ds \\ &+ \sum_{i=1}^d \int_0^t \int_{\mathbb{R}_0} \log(1-\theta_{i,s,x})\tilde{N}_i(ds,dx) \\ &+ \sum_{i=1}^d \int_0^t \int_{\mathbb{R}_0} (\log(1-\theta_{i,s,x}) + \theta_{i,s,x})\nu_i(dx)ds\Big),\end{aligned}$$

$t \in [0,T]$. *Define a measure* $\mathbb{P}^*$ *on* $\mathcal{F}_T$ *by*

$$d\mathbb{P}^*(\omega) = Z_T(\omega)d\mathbb{P}(\omega),$$

and we assume that $Z(T)$ *satisfies the Novikov condition, that is,*

$$\begin{aligned}\mathbb{E}\Big[\exp\Big(\frac{1}{2}\sum_{i=1}^d \int_0^T u_{i,s}^2 ds + \sum_{i=1}^d \int_0^T \int_{\mathbb{R}_0} \\ \Big\{(1-\theta_{i,s,x})\log(1-\theta_{i,s,x}) + \theta_{i,s,x}\Big\}\nu_i(dx)ds\Big)\Big] < \infty.\end{aligned}$$

Then $\mathbb{E}[Z_T] = 1$ *and hence* $\mathbb{P}^*$ *is a probability measure on* $\mathcal{F}_T$*. Furthermore, if we denote*

$$\tilde{N}_i^{\mathbb{P}^*}(dt,dx) := \theta_{i,t,x}\nu(dx)dt + \tilde{N}(dt,dx)$$

and

$$dW_{i,t}^{\mathbb{P}^*} := u_t dt + dW_{i,t},$$

then $\tilde{N}_i^{\mathbb{P}^*}(\cdot,\cdot)$ *and* $W_i^{\mathbb{P}^*}(\cdot)$ *are the compensated Poisson random measure of* $N_i(\cdot,\cdot)$ *and a standard Brownian motion under* $\mathbb{P}^*$*, respectively.*

5. A Clark-Ocone Type Formula under Change of Measure for Canonical Lévy Processes

5.1. A Clark-Ocone type formula under change of measure for canonical Lévy processes. In this section, we introduce a Clark-Ocone type formula under change of measure for canonical Lévy processes. Throughout this section, under the same setting as Theorem 4.2, we assume the following.

Assumption 1. (1) $u_i, u_i^2 \in \mathbb{L}_0^{j,1,2}$; and $2u_{i,s}D_{t,z}^j u_{i,s} + z(D_{t,z}^j u_{i,s})^2 \in L^2(q^j \times \mathbb{P})$ for a.e. $s \in [0,T], i,j = 1,\cdots,d$.
(2) $\theta_i + \log(1-\theta_i) \in \tilde{\mathbb{L}}_1^{j,1,2}$, and $\log(1-\theta_i) \in \mathbb{L}_1^{j,1,2}$, $i,j = 1,\cdots,d$.
(3) For q-a.e. $(s,x) \in [0,T]\times\mathbb{R}_0$, there is an $\varepsilon_{i,s,x} \in (0,1)$ such that $\theta_{i,s,x} < 1-\varepsilon_{i,s,x}, i = 1,\cdots,d$.
(4) $Z_T \in L^2(\mathbb{P})$; and $Z_T\{D_{t,0}^j \log Z_T \mathbf{1}_{\{0\}}(z) + \frac{e^{zD_{t,z}^j \log Z_T}-1}{z}\mathbf{1}_{\mathbb{R}_0}(z)\} \in L^2(q^j\times\mathbb{P})$.
(5) $F \in \mathbb{D}^{1,2}$ with $FZ_T \in L^2(\mathbb{P})$; and $Z_T D_{t,z}^j F + FD_{t,z}^j Z_T + zD_{t,z}^j F \cdot D_{t,z}^j Z_T \in L^2(q^j\times\mathbb{P}), j = 1,\cdots,d$.
(6) $FH_{t,z}^{j,*}, H_{t,z}^{j,*}D_{t,z}^j F \in L^1(\mathbb{P}^*), (t,z)$ -a.e. where $H_{t,z}^{j,*} = \exp(zD_{t,z}^j \log Z_T - \log(1-\theta_{j,t,z}))$.

To show the main theorem, we need the following:

Lemma 5.1. *We have*

$$\begin{aligned}&D^j_{t,0}Z_T\\&= Z_T\left[-\sigma_j^{-1}u_{j,t}-\sum_{i=1}^d\int_0^T D^j_{t,0}u_{i,s}dW^{\mathbb{P}^*}_{i,s}-\sum_{i=1}^d\int_0^T\int_{\mathbb{R}_0}\frac{D^j_{t,0}\theta_{i,s,x}}{1-\theta_{i,s,x}}\tilde{N}_i^{\mathbb{P}^*}(ds,dx)\right],\end{aligned} \tag{5.1}$$

for $j=1,\cdots,d$, q^j-a.e. $(t,z)\in[0,T]\times\{0\}$, $\mathbb{P}$-a.s. *and*

$$D^j_{t,z}Z_T = z^{-1}Z_T[\exp(zD^j_{t,z}\log Z_T)-1]\quad \text{for } q\text{−a.e. } (t,z)\in[0,T]\times\mathbb{R}_0,\ \mathbb{P}\text{−a.s.}, \tag{5.2}$$

where

$$\begin{aligned}D^j_{t,z}\log Z_T &= -\sum_{i=1}^d\int_0^T D^j_{t,z}u_{i,s}dW^{\mathbb{P}^*}_{i,s}-\sum_{i=1}^d\frac{1}{2}\int_0^T z(D^j_{t,z}u_{i,s})^2ds\\&\quad+\sum_{i=1}^d\int_0^T\int_{\mathbb{R}_0}\Big((1-\theta_{i,s,x})D^j_{t,z}\log(1-\theta_{i,s,x})+D^j_{t,z}\theta_{i,s,x}\Big)\nu_i(dx)ds\\&\quad+\sum_{i=1}^d\int_0^T\int_{\mathbb{R}_0}D^j_{t,z}\log(1-\theta_{i,s,x})\tilde{N}_i^{\mathbb{P}^*}(ds,dx)+z^{-1}\log(1-\theta_{j,t,z}),\end{aligned} \tag{5.3}$$

for q^j-a.e. $(t,z)\in[0,T]\times\mathbb{R}_0$, $j=1,\cdots,d$, $\mathbb{P}$-a.s.

Proof. By conditions (1), (2) and (3) in Assumption 1, Propositions 3.2, 3.3 and 3.4 imply $\log Z_T\in\mathbb{D}^{j,1,2}$. Moreover, from (4) in Assumption 1, Proposition 2.4 leads to $Z_T\in\mathbb{D}^{j,1,2}$,

$$\begin{aligned}D^j_{t,0}Z_T = Z_T\Bigg[&-D^j_{t,0}\sum_{i=1}^d\int_0^T u_{i,s}dW_{i,s}-\frac{1}{2}D^j_{t,0}\sum_{i=1}^d\int_0^T u^2_{i,s}ds\\&+D^j_{t,0}\sum_{i=1}^d\int_0^T\int_{\mathbb{R}_0}\log(1-\theta_{j,s,x})\tilde{N}_j(ds,dx)\\&+D^j_{t,0}\sum_{i=1}^d\int_0^T\int_{\mathbb{R}_0}(\log(1-\theta_{j,s,x})+\theta_{j,s,x})\nu_j(dx)ds\Bigg].\end{aligned} \tag{5.4}$$

and

$$D^j_{t,z}Z_T=\frac{\exp(\log Z_T+zD^j_{t,z}\log Z_T)-Z_T}{z}=z^{-1}Z_T[\exp(zD^j_{t,z}\log Z_T)-1].$$

We next calculate right side of (5.4). From assumption (1) in Assumption 1, Proposition 3.3 implies

$$D^j_{t,0}\sum_{i=1}^d\int_0^T u^2_{i,s}ds=\sum_{i=1}^d\int_0^T D^j_{t,0}u^2_{i,s}ds \tag{5.5}$$

and by Proposition 3.4,

$$\begin{aligned} & D^j_{t,0} \sum_{i=1}^d \int_0^T \int_{\mathbb{R}_0} (\log(1-\theta_{i,s,x}) + \theta_{i,s,x}) \nu(dx) ds \\ &= \sum_{i=1}^d \int_0^T \int_{\mathbb{R}_0} (D^j_{t,0} \log(1-\theta_{i,s,x}) + D^j_{t,0}\theta_{i,s,x}) \nu_i(dx) ds. \end{aligned} \tag{5.6}$$

Since condition (1) in Assumption 1 holds, by Corollary 2.5, we have

$$D^j_{t,0} u^2_{i,s} = 2u_{i,s} D^j_{t,0} u_s. \tag{5.7}$$

We calculate $D^j_{t,0} \log(1-\theta_{i,s,x})$. From (3) in Assumption 1, we have $\theta_{i,s,x} < 1-\varepsilon_{i,s,x}$. We fix $(s,x) \in [0,T] \times \mathbb{R}_0$. We denote

$$l_{i,s,x}(y) = -\varepsilon^{-1}_{i,s,x} y + \varepsilon^{-1}_{i,s,x} - 1 + \log \varepsilon_{i,s,x}$$

and

$$g_{i,s,x}(y) = \begin{cases} \log(1-y), & y < 1-\varepsilon_{i,s,x} \\ l_{s,x}(y), & y \geq 1-\varepsilon_{i,s,x} \end{cases}.$$

Then $g_{i,s,x} \in C^1_b(\mathbb{R})$ and

$$\log(1-\theta_{i,s,x}) = g_{i,s,x}(\theta_{i,s,x}).$$

Hence Proposition 2.4 implies that $\log(1-\theta_{i,s,x}) \in \mathbb{D}^{j,1,2}$ and

$$D^j_{t,0} \log(1-\theta_{i,s,x}) = D^j_{t,0} g_{i,s,x}(\theta_{i,s,x}) = g'_{i,s,x}(\theta_{i,s,x}) D^j_{t,0}\theta_{i,s,x} = -\frac{D^j_{t,0}\theta_{i,s,x}}{1-\theta_{i,s,x}}.$$

From condition (1), (2) in Assumption 1, Proposition 3.2 implies

$$D^j_{t,0} \sum_{i=1}^d \int_0^T u_{i,s} dW_{i,s} = \sigma_j^{-1} u_{j,t} + \sum_{i=1}^d \int_0^T D^j_{t,0} u_{i,s} dW_{i,s} \tag{5.8}$$

and

$$\begin{aligned} & D^j_{t,0} \sum_{i=1}^d \int_0^T \int_{\mathbb{R}_0} \log(1-\theta_{i,s,x}) \tilde{N}_i(ds,dx) \\ &= \sum_{i=1}^d \int_0^T \int_{\mathbb{R}_0} D^j_{t,0} \log(1-\theta_{i,s,x}) \tilde{N}_i(ds,dx). \end{aligned} \tag{5.9}$$

Hence, by (5.4) - (5.9), we obtain

$$
\begin{aligned}
&D^j_{t,0} Z_T \\
&= Z_T \Bigg[-\sigma_j^{-1} u_{j,t} - \sum_{i=1}^d \int_0^T D^j_{t,0} u_{i,s} dW_{i,s} - \sum_{i=1}^d \int_0^T u_{i,s} D^j_{t,0} u_{i,s} ds \\
&\quad - \sum_{i=1}^d \int_0^T \int_{\mathbb{R}_0} \frac{D^j_{t,0}\theta_{i,s,x}}{1-\theta_{i,s,x}} \tilde{N}_i(ds,dx) \\
&\quad + \sum_{i=1}^d \int_0^T \int_{\mathbb{R}_0} \left(-\frac{D^j_{t,0}\theta_{i,s,x}}{1-\theta_{i,s,x}} + D^j_{t,0}\theta_{i,s,x} \right) \nu_i(dx) ds \Bigg] \\
&= Z_T \Bigg[-\sigma_j^{-1} u_{j,t} - \sum_{i=1}^d \int_0^T D^j_{t,0} u_{i,s} dW^{\mathbb{P}^*}_{i,s} - \sum_{i=1}^d \int_0^T \int_{\mathbb{R}_0} \frac{D^j_{t,0}\theta_{i,s,x}}{1-\theta_{i,s,x}} \tilde{N}^{\mathbb{P}^*}_i(ds,dx) \Bigg].
\end{aligned}
$$

We next calculate $D_{t,z} \log Z_T$. By conditions (1) and (2) in Assumption 1, Proposition 3.2, Proposition 3.3 and Proposition 3.4 show that

$$
\begin{aligned}
&D^j_{t,z} \log Z_T \\
&= -D^j_{t,z} \sum_{i=1}^d \int_0^T u_{i,s} dW_{i,s} - \frac{1}{2} \sum_{i=1}^d D^j_{t,z} \int_0^T u^2_{i,s} ds \\
&\quad + D^j_{t,z} \sum_{i=1}^d \int_0^T \int_{\mathbb{R}_0} x^{-1} \log(1-\theta_{i,s,x}) x \tilde{N}_i(ds,dx) \\
&\quad + D^j_{t,z} \sum_{i=1}^d \int_0^T \int_{\mathbb{R}_0} (\log(1-\theta_{i,s,x}) + \theta_{i,s,x}) \nu_i(dx) ds \\
&= -\sum_{i=1}^d \int_0^T D^j_{t,z} u_{i,s} dW_{i,s} - \frac{1}{2} \sum_{i=1}^d \int_0^T D^j_{t,z} (u_{i,s})^2 ds \\
&\quad + \sum_{i=1}^d \int_0^T \int_{\mathbb{R}_0} D^j_{t,z} \log(1-\theta_{i,s,x}) \tilde{N}_i(ds,dx) \\
&\quad + \int_0^T \int_{\mathbb{R}_0} \left(D^j_{t,z} \log(1-\theta_{i,s,x}) + D^j_{t,z}\theta_{i,s,x} \right) \nu_i(dx) ds \\
&\quad + \frac{\log(1-\theta_{j,t,z})}{z}.
\end{aligned} \tag{5.10}
$$

Now we calculate $D^j_{t,z}(u_{i,s})^2$. Corollary 2.5 implies

$$
D^j_{t,z}(u_{i,s})^2 = 2u_{i,s} D^j_{t,z} u_{i,s} + z(D^j_{t,z} u_{i,s})^2, \tag{5.11}
$$

because, $u^i \in \mathbb{D}^{j,1,2}$ and condition (1) in Assumption 1 hold. From equations (5.10) and (5.11), we have

$$\begin{aligned} D^j_{t,z}\log Z_T = &-\sum_{i=1}^d \int_0^T D^j_{t,z}u_{i,s}dW^{\mathbb{P}^*}_{i,s} - \sum_{i=1}^d \frac{1}{2}\int_0^T z(D^j_{t,z}u_{i,s})^2 ds \\ &+ \sum_{i=1}^d \int_0^T \int_{\mathbb{R}_0} \Big((1-\theta_{i,s,x})D^j_{t,z}\log(1-\theta_{i,s,x}) + D^j_{t,z}\theta_{i,s,x}\Big)\nu_i(dx)ds \\ &+ \sum_{i=1}^d \int_0^T \int_{\mathbb{R}_0} D^j_{t,z}\log(1-\theta_{i,s,x})\tilde{N}^{\mathbb{P}^*}_i(ds,dx) + z^{-1}\log(1-\theta_{j,t,z}). \end{aligned}$$

□

We next introduce a Clark-Ocone type formula under change of measure for canonical multidimensional Lévy processes.

Theorem 5.2.

$$\begin{aligned} F = \mathbb{E}_{\mathbb{P}^*}[F] &+ \sum_{j=1}^d \sigma_j \int_0^T \mathbb{E}_{\mathbb{P}^*}\left[D^j_{t,0}F - FK^j_t \Big| \mathcal{F}_{t-}\right] dW^{\mathbb{P}^*}_{j,t} \\ &+ \sum_{j=1}^d \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[F(H^{j,*}_{t,z}-1) + zH^{j,*}_{t,z}D^j_{t,z}F|\mathcal{F}_{t-}]\tilde{N}^{\mathbb{P}^*}_j(dt,dz), a.s. \end{aligned}$$

holds, where

$$K^j_t = \sum_{i=1}^d \int_0^T D^j_{t,0}u_{i,s}dW^{\mathbb{P}^*}_{i,s} + \sum_{i=1}^d \int_0^T \int_{\mathbb{R}_0} \frac{D^j_{t,0}\theta_{i,s,x}}{1-\theta_{i,s,x}}\tilde{N}^{\mathbb{P}^*}_i(ds,dx).$$

Proof. First we denote $\Lambda_t := Z_t^{-1} = e^{-\log Z_t}, t \in [0,T]$. Then by the Itô formula (see, e.g., Theorem 9.5 of Di Nunno et al. [6]), we have

$$\begin{aligned} d\Lambda_t &= \Lambda_{t-}\sum_{i=1}^d \left(\frac{1}{2}u^2_{i,t} - \int_{\mathbb{R}_0}(\log(1-\theta_{i,t,z}) + \theta_{i,t,z})\nu_i(dz)\right)dt + \frac{1}{2}\Lambda_{t-}\sum_{i=1}^d u^2_{i,t}dt \\ &\quad + \Lambda_{t-}\sum_{i=1}^d u_{i,t}dW_{i,t} + \sum_{i=1}^d \int_{\mathbb{R}_0}\Lambda_{t-}\left(\frac{1}{1-\theta_{i,t,z}} - 1\right)\tilde{N}_i(dt,dz) \\ &\quad + \sum_{i=1}^d \int_{\mathbb{R}_0}\left[\Lambda_{t-}\cdot\frac{1}{1-\theta_{i,t,z}} - \Lambda_{t-} + \Lambda_{t-}\log(1-\theta_{i,t,z})\right]\nu_i(dz)dt \\ &= \Lambda_{t-}\sum_{i=1}^d \left[u^2_{,it}dt + u_{i,t}dW_{i,t} + \int_{\mathbb{R}_0}\frac{\theta^2_{i,t,z}}{1-\theta_{i,t,z}}\nu_i(dz)dt \right. \\ &\qquad\qquad \left. + \int_{\mathbb{R}_0}\frac{\theta_{i,t,z}}{1-\theta_{i,t,z}}\tilde{N}_i(dt,dz)\right] \\ &= \Lambda_{t-}\sum_{i=1}^d \left[u_{i,t}dW^{\mathbb{P}^*}_{i,t} + \int_{\mathbb{R}_0}\frac{\theta_{i,t,z}}{1-\theta_{i,t,z}}\tilde{N}^{\mathbb{P}^*}_i(dt,dz)\right]. \end{aligned}$$

Denoting $Y_t := \mathbb{E}_{\mathbb{P}^*}[F|\mathcal{F}_t], t \in [0,T]$, we have $Y_t = \Lambda_t \mathbb{E}[Z_T F|\mathcal{F}_t]$ by condition (5) in Assumption 1 and the Beyes rule (see, e.g., Lemma 4.7 of Di Nunno et al. [6]). From (5) in Assumption 1, Corollary 2.5 implies that $Z_T F \in \mathbb{D}^{1,2}$. Hence we apply Proposition 4.1 to $Z_T F$ and take conditional expectation, we have

$$\mathbb{E}[Z_T F|\mathcal{F}_t] = \mathbb{E}[Z_T F] + \sum_{j=1}^{d} \int_0^t \int_{\mathbb{R}} \mathbb{E}[D^j_{s,z}(Z_T F)|\mathcal{F}_{s-}] Q_j(ds, dz).$$

Denoting $V_t := \mathbb{E}[Z_T F|\mathcal{F}_t]$, we have $Y_t = \Lambda_t V_t$. Itô's product rule implies that

$$\begin{aligned}
dY_t &= \Lambda_{t-} dV_t + V_{t-} d\Lambda_t + d[\Lambda, V]_t \\
&= \Lambda_{t-} \sum_{j=1}^{d} [\sigma_j \mathbb{E}[D^j_{t,0}(Z_T F)|\mathcal{F}_{t-}] dW_{j,t} + \int_{\mathbb{R}_0} \mathbb{E}[D^j_{t,z}(Z_T F)|\mathcal{F}_{t-}] z \tilde{N}_j(dt, dz)] \\
&\quad + V_{t-}\Lambda_{t-} \sum_{j=1}^{d} \left[u_{j,t} dW_t^{\mathbb{P}^*} + \int_{\mathbb{R}_0} \frac{\theta_{j,t,z}}{1-\theta_{j,t,z}} \tilde{N}_j^{\mathbb{P}^*}(dt, dz) \right] \\
&\quad + \Lambda_{t-} \sum_{j=1}^{d} \Bigg[\sigma_j u_{j,t} \mathbb{E}[D^j_{t,0}(Z_T F)|\mathcal{F}_{t-}] \\
&\qquad + \int_{\mathbb{R}_0} \frac{\theta_{j,t,z}}{1-\theta_{j,t,z}} \mathbb{E}[D^j_{t,z}(Z_T F)|\mathcal{F}_{t-}] z \nu_j(dz) \Bigg] dt \\
&\quad + \Lambda_{t-} \sum_{j=1}^{d} \int_{\mathbb{R}_0} \frac{\theta_{j,t,z}}{1-\theta_{j,t,z}} \mathbb{E}[D^j_{t,z}(Z_T F)|\mathcal{F}_{t-}] z \tilde{N}_j(ds, dz) \\
&= \Lambda_{t-} \sum_{j=1}^{d} \mathbb{E}[\sigma_j D^j_{t,0}(Z_T F)|\mathcal{F}_{t-}] dW_t^{\mathbb{P}^*} + \Lambda_{t-} \sum_{j=1}^{d} \mathbb{E}[Z_T F u_{j,t}|\mathcal{F}_{t-}] dW_{j,t}^{\mathbb{P}^*} \\
&\quad + \Lambda_{t-} \sum_{j=1}^{d} \int_{\mathbb{R}_0} \frac{\mathbb{E}[D^j_{t,z}(Z_T F)|\mathcal{F}_{t-}]}{1-\theta_{j,t,z}} z \tilde{N}_j^{\mathbb{P}^*}(dt, dz) \\
&\quad + \Lambda_{t-} \sum_{j=1}^{d} \int_{\mathbb{R}_0} \mathbb{E}\left[Z_T F \frac{\theta_{j,t,z}}{1-\theta_{j,t,z}} \middle| \mathcal{F}_{t-} \right] \tilde{N}_j^{\mathbb{P}^*}(dt, dz).
\end{aligned} \tag{5.12}$$

Now we shall calculate $D_{t,0}(Z_T F)$ and $D_{t,z}(Z_T F)$. As for $D_{t,0}(Z_T F)$, by (5) in Assumption 1, Corollary 2.5 yields that

$$D_{t,0}(Z_T F) = F D_{t,0} Z_T + Z_T D_{t,0} F. \tag{5.13}$$

Therefore combining (5.13) with (5.1), we can conclude

$$\begin{aligned}
&D^j_{t,0}(Z_T F) \\
&= F D^j_{t,0} Z_T + Z_T D^j_{t,0} F \\
&= F Z_T \left[-\sigma_j^{-1} u_{j,t} - \sum_{i=1}^{d} \int_0^T D^j_{t,0} u_{i,s} dW_{i,s}^{\mathbb{P}^*} - \sum_{i=1}^{d} \int_0^T \int_{\mathbb{R}_0} \frac{D^j_{t,0}\theta_{i,s,x}}{1-\theta_{i,s,x}} \tilde{N}_i^{\mathbb{P}^*}(ds, dx) \right] \\
&\quad + Z_T D^j_{t,0} F
\end{aligned}$$

$$= Z_T \left[D^j_{t,0} F - F \left(\sigma_j^{-1} u_{j,t} + K^j_t \right) \right]. \tag{5.14}$$

Next we calculate $D^j_{t,z}(Z_T F)$. From condition (5), Corollary 2.5 implies that

$$D^j_{t,z}(Z_T F) = F D^j_{t,z} Z_T + Z_T D^j_{t,z} F + z D^j_{t,z} Z_T \cdot D^j_{t,z} F. \tag{5.15}$$

From (5.2),

$$D^j_{t,z} Z_T = z^{-1} Z_T [(1 - \theta_{j,t,z}) H^{j,*}_{t,z} - 1]. \tag{5.16}$$

Therefore, combining (5.15) and (5.16), we obtain

$$\begin{aligned}
&D^j_{t,z}(Z_T F) \\
&= z^{-1} Z_T [(1 - \theta_{j,t,z}) H^{j,*}_{t,z} - 1] F + Z_T D^j_{t,z} F + Z_T [(1 - \theta_{j,t,z}) H^{*j,}_{t,z} - 1] D^j_{t,z} F \\
&= Z_T \left[z^{-1}((1 - \theta_{j,t,z}) H^{j,*}_{t,z} - 1) F + (1 - \theta_{j,t,z}) H^{j,*}_{t,z} D^j_{t,z} F \right].
\end{aligned} \tag{5.17}$$

From (5.12), (5.14), (5.17), we arrive at:

$$\begin{aligned}
dY_t &= \Lambda_{t-} \sum_{j=1}^d \mathbb{E}\left[Z_T \left[\sigma_j D^j_{t,0} F - F \left(u_{j,t} + \sigma_j K^j_t \right) \right] \middle| \mathcal{F}_{t-} \right] dW^{\mathbb{P}^*}_{j,t} \\
&\quad + \Lambda_{t-} \sum_{j=1}^d \int_{\mathbb{R}_0} \mathbb{E}\left[Z_T \left[F \left(H^{j,*}_{t,z} - \frac{1}{1 - \theta_{j,t,z}} \right) + z H^{j,*}_{t,z} D^j_{t,z} F \right] \middle| \mathcal{F}_{t-} \right] \tilde{N}^{\mathbb{P}^*}(dt, dz) \\
&\quad + \Lambda_{t-} \sum_{j=1}^d \mathbb{E}[Z_T F u_{j,t} | \mathcal{F}_{t-}] dW^{\mathbb{P}^*}_{j,t} \\
&\quad + \Lambda_{t-} \sum_{j=1}^d \int_{\mathbb{R}_0} \mathbb{E}\left[Z_T F \frac{\theta_{j,t,z}}{1 - \theta_{j,t,z}} \middle| \mathcal{F}_{t-} \right] \tilde{N}^{\mathbb{P}^*}_j(dt, dz) \\
&= \sum_{j=1}^d \sigma_j \Lambda_{t-} \mathbb{E}\left[Z_T \left[D^j_{t,0} F - F K^j_t \right] \middle| \mathcal{F}_{t-} \right] dW^{\mathbb{P}^*}_{j,t} \\
&\quad + \sum_{j=1}^d \Lambda_{t-} \int_{\mathbb{R}_0} \mathbb{E}\left[Z_T \{ F \left(H^{j,*}_{t,z} - 1 \right) + z H^{j,*}_{t,z} D^j_{t,z} F \} \middle| \mathcal{F}_{t-} \right] \tilde{N}^{\mathbb{P}^*}(dt, dz).
\end{aligned}$$

From (1) and (2) in Assumption 1, we have $K^j_t \in L^2(\mathbb{P})$ t-a.e. Hence, by (5) in Assumption 1,

$$\mathbb{E}_{\mathbb{P}^*}[|F K^j_t|] = \mathbb{E}[|F K^j_t | Z_T] \le (\mathbb{E}[|K^j_t|^2])^{1/2} (\mathbb{E}[|F Z_T|^2])^{1/2} < \infty.$$

Moreover, from (5) in Assumption 1, we have $D^j_{t,0} F \in L^2(\mathbb{P})$ t-a.e. and

$$\mathbb{E}_{\mathbb{P}^*}[|D^j_{t,0} F|] = \mathbb{E}[|D^j_{t,0} F | Z_T] \le (\mathbb{E}[|D^j_{t,0} F|^2])^{1/2} (\mathbb{E}[Z_T^2])^{1/2} < \infty.$$

Then by (6) in Assumption 1 and $F, D^j_{t,0}F, FK^j_t \in L^1(\mathbb{P}^*)$ t-a.e., the Beyes rule implies

$$
\begin{aligned}
dY_t = &\sum_{j=1}^d \sigma_j \mathbb{E}_{\mathbb{P}^*}\left[D^j_{t,0}F - FK^j_t \middle| \mathcal{F}_{t-}\right] dW^{\mathbb{P}^*}_{j,t} \\
&+ \sum_{j=1}^d \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[F(H^{j,*}_{t,z} - 1) + zH^{j,*}_{t,z} D^j_{t,z}F|\mathcal{F}_{t-}]\tilde{N}^{\mathbb{P}^*}_j(dt, dz). \qquad (5.18)
\end{aligned}
$$

Since $Y_t = \mathbb{E}_{\mathbb{P}^*}[F|\mathcal{F}_T] = F, Y(0) = \mathbb{E}_{\mathbb{P}^*}[F|\mathcal{F}_0] = \mathbb{E}_{\mathbb{P}^*}[F]$, Integrating equation (5.18) gives

$$
\begin{aligned}
F - \mathbb{E}_{\mathbb{P}^*}[F] = &\sum_{j=1}^d \sigma_j \int_0^T \mathbb{E}_{\mathbb{P}^*}\left[D^j_{t,0}F - FK^j_t \middle| \mathcal{F}_{t-}\right] dW^{\mathbb{P}^*}_{j,t} \\
&+ \sum_{j=1}^d \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[F(H^{j,*}_{t,z} - 1) + zH^{j,*}_{t,z} D^j_{t,z}F|\mathcal{F}_{t-}]\tilde{N}^{\mathbb{P}^*}_j(dt, dz).
\end{aligned}
$$

The proof is concluded. □

Corollary 5.3. *Assume in addition to all assumptions of Theorem 5.2, that u and θ are deterministic functions, then we have*

$$
\begin{aligned}
F = \mathbb{E}_{\mathbb{P}^*}[F] &+ \sum_{j=1}^d \sigma_j \int_0^T \mathbb{E}_{\mathbb{P}^*}[D^j_{t,0}F|\mathcal{F}_{t-}]dW^{\mathbb{P}^*}_{j,t} \\
&+ \sum_{j=1}^d \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[D^j_{t,z}F|\mathcal{F}_{t-}]z\tilde{N}^{\mathbb{P}^*}_j(dt, dz).
\end{aligned}
$$

Proof. If u and θ are deterministic functions, then we have $D^j_{t,z}u_{i,s} = 0 = D^j_{t,z}\theta_{i,s,x}$ and $H^{*j,}(t, z) = 1$ for $i, j = 1, \cdots, d$. Therefore, thanks to Theorem 5.2, we can get the claimed equation. □

Remark 5.4. (1) If $F \in \mathbb{D}^{1,2}$, $u \equiv 0$ and $\theta \equiv 0$,, then we can see that assumptions of Theorem 4.2 and Assumption 1 hold and we obtain equation (4.1).
(2) If $d = 1$, we obtain Theorem 4.4 and Corollary 4.8 in [14].

6. Local Risk Minimization for Lévy Markets

6.1. Model description. We consider a financial market being composed of one risk-free asset and $d \geq 1$ risky assets with finite time horizon T. For simplicity, we assume that the interest rate of the market is given by 0, that is, the price of the risk-free asset is 1 at all times. The fluctuations of the risky assets $S = (S^1, \cdots, S^d)^T$ are assumed to be given by solutions to the following stochastic differential equations (SDE, for short) on canonical space $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t\in[0,T]})$:

$$
dS^i_t = S^i_{t-}\left[\alpha^i_t dt + \beta_{i,t} dW_{i,t} + \int_{\mathbb{R}_0} \gamma_{i,t,z}\tilde{N}_i(dt, dz)\right], \quad S^i_0 > 0, i = 1, \cdots, d, \quad (6.1)
$$

where α, β and γ are predictable processes. Recall that γ is a stochastic process measurable with respect to the σ-algebra generated by $A \times (s,u] \times B$, $A \in \mathcal{F}_s$, $0 \leq s < u \leq T$, $B \in \mathcal{B}(\mathbb{R}_0)$. Now, we assume the following:

Assumption 2. (1) (6.1) has a solution S satisfying the so-called structure condition (SC, for short). That is, S is a special semimartingale with the canonical decomposition $S = S_0 + M + A$ such that

$$\sum_{i=1}^{d} \left\| [M^i]_T^{1/2} + \int_0^T |dA_s^i| \right\|_{L^2(\mathbb{P})} < \infty, \tag{6.2}$$

where $M = (M^1, \cdots, M^d)^T$, $A = (A^1, \cdots, A^d)^T$,
$dM_t^i = S_{t-}^i(\beta_{i,t} dW_{i,t} + \int_{\mathbb{R}_0} \gamma_{i,t,z} \tilde{N}_i(dt,dz))$ and $dA_t^i = S_{t-}^i \alpha_t^i dt$ for $i = 1, \cdots, d$. Moreover, defining a process

$$\lambda_t^i := \frac{\alpha_t^i}{S_{t-}^i(\beta_{i,t}^2 + \int_{\mathbb{R}_0} \gamma_{i,t,z}^2 \nu_i(dz))},$$

we have $A^i = \int \lambda d\langle M^i \rangle$. Thirdly, the mean-variance trade-off process $K_t^i := \int_0^t \lambda_s^2 d\langle M^i \rangle_s$ is finite, that is, K_T^i is finite $\mathbb{P}$-a.s.

(2) $\gamma_{i,t,z} > -1$, (t,z,ω)-a.e. for $i = 1, \cdots, d$, that is,

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} \mathbf{1}_{\{\gamma_{i,t,z} \leq -1\}} \nu_j(dz) dt\right] = 0.$$

Remark 6.1. (1) The SC is closely related to the no-arbitrage condition. For more details on the SC, see Schweizer [11] and [12].
(2) The process K as well as A is continuous.
(3) (6.2) implies that $\sup_{t \in [0,T]} |S_t| \in L^2(\mathbb{P})$ by Theorem V.2 of Protter [10].
(4) Condition 2 ensures that $S_t > 0$ for any $t \in [0,T]$.

6.2. Locally risk-minimizing. We define locally risk-minimizing (LRM, for short) for a contingent claim $F \in L^2(\mathbb{P})$. The following definition is based on Theorem 1.6 of Schweizer [12].

Definition 6.2. (1) Θ_S denotes the space of all $\mathbb{R}$-valued predictable processes $\xi = (\xi^1, \cdots, \xi^d)^T$ satisfying

$$\mathbb{E}\left[\sum_{i=1}^{d} \int_0^T (\xi_t^i)^2 d\langle M^i \rangle_t + \Big(\sum_{i=1}^{d} \int_0^T |\xi_t^i dA_t^i|\Big)^2\right] < \infty.$$

(2) An L^2-*strategy* is given by $\varphi = (\xi, \eta)$, where $\xi \in \Theta_S$ and η is an adapted process such that $V(\varphi) := \xi S + \eta = \sum_{i=1}^d (\xi^i) S^i + \eta$ is a right continuous process with $\mathbb{E}[V_t^2(\varphi)] < \infty$ for every $t \in [0,T]$. Note that ξ_t^i (resp. η_t) represents the amount of units of the risky asset S^i (resp. the risk-free asset) an investor holds at time t.
(3) For $F \in L^2(\mathbb{P})$, the process $C^F(\varphi)$ defined by $C_t^F(\varphi) := F 1_{\{t=T\}} + V_t(\varphi) - \sum_{i=1}^d \int_0^t \xi_s^i dS_s^i$ is called the *cost process* of $\varphi = (\xi, \eta)$ for F.
(4) An L^2-strategy φ is said locally risk-minimizing for F if $V_T(\varphi) = 0$ and $C^F(\varphi)$

is a martingale orthogonal to M, that is, $[C^F(\varphi), M]$ is a uniformly integrable martingale.

The above definition of LRM is a simplified version, since the original one, introduced in Schweizer [11] and [12], is rather complicated

Now, we focus on a representation of LRM. To this end, we define Föllmer-Schweizer decomposition (FS decomposition, for short).

Definition 6.3. An $F \in L^2(\mathbb{P})$ admits a *Föllmer-Schweizer decomposition* if it can be described by

$$F = F_0 + \int_0^T \xi_t^F dS_t + L_T^F, \tag{6.3}$$

where $F_0 \in \mathbb{R}$, $\xi^F \in \Theta_S$ and L^F is a square-integrable martingale orthogonal to M with $L_0^F = 0$.

Proposition 5.2 of Schweizer [12] shows the following:

Proposition 6.4 (Proposition 5.2 of Schweizer [12])**.** *Under Assumption 2, an LRM $\varphi = (\xi, \eta)$ for F exists if and only if F admits an FS decomposition; and its relationship is given by*

$$\xi_t = \xi_t^F, \quad \eta_t = F_0 + \int_0^t \xi_s^F dS_s + L_t^F - F1_{\{t=T\}} - \xi_t^F S_t.$$

As a result, it suffices to obtain a representation of ξ^F in (6.3) in order to obtain LRM. Henceforth, we identify ξ^F with LRM. To this end, we consider the process $Z := \mathcal{E}(-\int \lambda dM)$, where $\mathcal{E}(Y)$ represents the stochastic exponential of Y, that is, Z is a solution to the SDE $dZ_t = -\lambda_t Z_{t-} dM_t$. In addition to Assumption 2, we suppose the following:

Assumption 3. Z is a positive square integrable martingale; and $Z_T F \in L^2(\mathbb{P})$.

Definition 6.5. A martingale measure $\mathbb{P}^* \sim \mathbb{P}$ is called *minimal* if any square-integrable $\mathbb{P}$-martingale orthogonal to M remains a martingale under $\mathbb{P}^*$.

We can see the following:

Lemma 6.6. *Under Assumption 2, if Z is a positive square integrable martingale, then a minimal martingale measure $\mathbb{P}^*$ exists with $d\mathbb{P}^* = Z_T d\mathbb{P}$.*

Proof. Since $d(ZS) = S_- dZ + Z_- dM + Z_- \lambda d\langle M\rangle - Z_- \lambda d[M]$, the product process ZS is a $\mathbb{P}$-local martingale. So that, defining a probability measure $\mathbb{P}^*$ as $d\mathbb{P}^* = Z_T d\mathbb{P}$, we have that S is a $\mathbb{P}^*$-martingale, since $\sup_{t\in[0,T]} |S_t|$ and Z_T are in $L^2(\mathbb{P})$. Next, for any L a square-integrable $\mathbb{P}$-martingale with null at 0 orthogonal to M, LZ is a $\mathbb{P}$-local martingale. By the square integrability of L, L remains a martingale under $\mathbb{P}^*$. Thus, $\mathbb{P}^*$ is a minimal martingale measure. $\square$

7. Representation Results for LRM

In this section, we focus on representations of LRM ξ^F for claim F. First of all, we study it through the martingale representation theorem.

7.1. Approach based on the martingale representation theorem. Throughout this subsection, we assume Assumptions 2 and 3. Let $\mathbb{P}^*$ be a minimal martingale measure, that is, $d\mathbb{P}^* = Z_T d\mathbb{P}$ holds. The martingale representation theorem (see, e.g. section 2 of Benth et al. [4]) provides

$$Z_T F = \mathbb{E}_{\mathbb{P}^*}[F] + \sum_{i=1}^d \int_0^T g_t^{i,0} dW_{i,t} + \sum_{i=1}^d \int_0^T \int_{\mathbb{R}_0} g_{t,z}^{i,1} \tilde{N}_i(dt,dz)$$

for some predictable processes $g_t^{i,0}$ and $g_{t,z}^{i,1}$, $1 \le i \le d$. By the same sort of calculations as the proof of Theorem 5.2, we have

$$\begin{aligned} F &= \mathbb{E}_{\mathbb{P}^*}[F] + \sum_{i=1}^d \int_0^T \frac{g_t^{i,0} + \mathbb{E}[Z_T F|\mathcal{F}_{t-}]u_{i,t}}{Z_{t-}} dW_{i,t}^{\mathbb{P}^*} \\ &\quad + \sum_{i=1}^d \int_0^T \int_{\mathbb{R}_0} \frac{g_{t,z}^{i,1} + \mathbb{E}[Z_T F|\mathcal{F}_{t-}]\theta_{i,t,z}}{Z_{t-}(1-\theta_{i,t,z})} \tilde{N}_i^{\mathbb{P}}(dt,dz) \\ &=: \mathbb{E}_{\mathbb{P}^*}[F] + \sum_{i=1}^d \int_0^T h_t^{i,0} dW_{i,t}^{\mathbb{P}^*} + \sum_{i=1}^d \int_0^T \int_{\mathbb{R}_0} h_{t,z}^{i,1} \tilde{N}_i^{\mathbb{P}^*}(dt,dz) \end{aligned}$$

where $u_{i,t} := \lambda_t^i S_{t-}^i \beta_{i,t}$, $\theta_{i,t,z} := \lambda_t^i S_{t-}^i \gamma_{i,t,z}$,

$$dW_{i,t}^{\mathbb{P}^*} := dW_{i,t} + u_{i,t} dt$$

and

$$\tilde{N}^{\mathbb{P}_i^*}(dt,dz) := \tilde{N}_i(dt,dz) + \theta_{i,t,z}\nu_i(dz)dt.$$

Girsanov's theorem implies that $W_i^{\mathbb{P}^*}$ and $\tilde{N}_i^{\mathbb{P}^*}$ are Brownian motions and the compensated Poisson random measures of N_i under $\mathbb{P}^*$, respectively. Additionally, we assume that

$$\sum_{i=1}^d \mathbb{E}\left[\int_0^T \left\{(h_t^{i,0})^2 + \int_{\mathbb{R}_0} (h_{t,z}^{i,1})^2 \nu_i(dz)\right\} dt\right] < \infty. \tag{7.1}$$

Denoting $i_t^{j,0} := h_t^{j,0} - \xi_t^i S_{t-}^i \beta_{i,t}$, $i_{t,z}^{i,1} := h_{t,z}^{i,1} - \xi_t^i S_{t-}^i \gamma_{i,t,z}$, and

$$\xi_t^i := \frac{\lambda_t^i}{\alpha_t^i} \{h_t^{i,0}\beta_{i,t} + \int_{\mathbb{R}_0} h_{t,z}^{i,1}\gamma_{i,t,z}\nu_i(dz))\}, \tag{7.2}$$

we can see

$$i_t^{i,0}\beta_{i,t} + \int_{\mathbb{R}_0} i_{t,z}^{j,1}\gamma_{i,t,z}\nu_i(dz) = 0$$

for any $t \in [0,T]$, which implies $i_t^{i,0}u_{i,t} + \int_{\mathbb{R}_0} i_{t,z}^{i,1}\theta_{i,t,z}\nu_i(dz) = 0$. We have then

$$\begin{aligned} F - \mathbb{E}_{\mathbb{P}}[F] - \int_0^T \xi_t dS_t &= \sum_{i=1}^d \int_0^T i_t^{i,0} dW_{i,t}^{\mathbb{P}} + \sum_{i=1}^d \int_0^T \int_{\mathbb{R}_0} i_{t,z}^{i,1} \tilde{N}_i^{\mathbb{P}^*}(dt,dz) \\ &= \sum_{i=1}^d \int_0^T i_t^{i,0} dW_{i,t} + \sum_{i=1}^d \int_0^T \int_{\mathbb{R}_0} i_{t,z}^{i,1} \tilde{N}_i(dt,dz). \end{aligned}$$

The following lemma implies that $L_t^F := \mathbb{E}[F - \mathbb{E}_{\mathbb{P}^*}[F] - \int_0^T \xi_s dS_s | \mathcal{F}_t]$ is a square integrable martingale orthogonal to M with $L_0^F = 0$.

Lemma 7.1. *Under Assumptions 2 and 3, and (7.1), we have*

$$\sum_{i=1}^d \mathbb{E}\left[\int_0^T (i_t^{i,0})^2 dt + \int_0^T \int_{\mathbb{R}_0} (i_{t,z}^{i,1})^2 \nu_i(dz) dt\right] < \infty.$$

Proof. Noting that $\frac{\beta_{i,t}^2}{\beta_{i,t}^2 + \int_{\mathbb{R}_0} \gamma_{i,t,x}^2 \nu_i(dx)}$ and $\frac{\int_{\mathbb{R}_0} \gamma_{i,t,x}^2 \nu_i(dx)}{\beta_{i,t}^2 + \int_{\mathbb{R}_0} \gamma_{i,t,x}^2 \nu_i(dx)}$ are less than 1, we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \xi_{i,t}^2 (S_{t-}^i)^2 \beta_{i,t}^2 dt\right] \\
&\le 2\mathbb{E}\left[\int_0^T \frac{\beta_{i,t}^4 (h_t^{i,0})^2 + \beta_{i,t}^2 \left(\int_{\mathbb{R}_0} h_{t,x}^{i,1} \gamma_{i,t,x} \nu_i(dx)\right)^2}{\left(\beta_{i,t}^2 + \int_{\mathbb{R}_0} \gamma_{i,t,x}^2 \nu_i(dx)\right)^2} dt\right] \\
&\le 2\mathbb{E}\left[\int_0^T \frac{\beta_{i,t}^4 (h_t^{i,0})^2 + \beta_{i,t}^2 \int_{\mathbb{R}_0} (h_{t,x}^{i,1})^2 \nu_i(dx) \int_{\mathbb{R}_0} \gamma_{i,t,x}^2 \nu_i(dx)}{\left(\beta_{i,t}^2 + \int_{\mathbb{R}_0} \gamma_{i,t,x}^2 \nu_i(dx)\right)^2} dt\right] \\
&\le 2\mathbb{E}\left[\int_0^T \left\{(h_t^{i,0})^2 + \int_{\mathbb{R}_0} (h_{t,z}^{i,1})^2 \nu_i(dz)\right\} dt\right].
\end{aligned}$$

By the same way as the above, we can see $\mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} (\xi_t^i)^2 (S_{t-}^i)^2 \gamma_{i,t,z}^2 \nu_i(dz) dt\right] < \infty$. Together with (7.1), Lemma 7.1 follows. □

Theorem 7.2. *Assume that Assumptions 2, 3, and (7.1). We have then $\xi_j^F = \xi^j$, $1 \le j \le d$ defined in (7.2).*

In the above theorem, a representation of LRM ξ^F is obtained under a mild setting. Since the processes $h^{j,0}$ and $h^{j,1}$ appeared in (7.2) are induced by the martingale representation theorem, it is almost impossible to calculate them explicitly, and confirm if (7.1) holds. In the rest of this section, we aim to get concrete expressions for $h^{j,0}$ and $h^{j,1}$ by using Malliavin calculus.

7.2. Main results of LRM. We now calculate h^0 and h^1 by using Theorem 5.2. Together with Theorem 7.2, we obtain the following:

Theorem 7.3. *Under Assumptions 2, 3 and 1, $h^{j,0}$ and $h^{j,1}$, $1 \le j \le d$ are described as*

$$h_t^{j,0} = \sigma_j \mathbb{E}_{\mathbb{P}^*}\left[D_{t,0}^j F - F \sum_{i=1}^d \left[\int_0^T D_{t,0}^j u_{i,s} dW_{i,s}^{\mathbb{P}^*} + \int_0^T \int_{\mathbb{R}_0} \frac{D_{t,0}^j \theta_{i,s,x}}{1 - \theta_{i,s,x}} \tilde{N}_i^{\mathbb{P}^*}(ds, dx)\right] \Big| \mathcal{F}_{t-}\right], \tag{7.3}$$

$$h_{t,z}^{j,1} = \mathbb{E}_{\mathbb{P}^*}[F(H_{t,z}^{j,*} - 1) + z H_{t,z}^{j,*} D_{t,z}^j F | \mathcal{F}_{t-}]. \tag{7.4}$$

Moreover, LRM $\xi^F = (\xi^{1,F}, \cdots \xi^{d,F})^T$ are given by substituting (7.3) and (7.4) for $h^{j,0}$ and $h^{j,1}$, $1 \le j \le d$ in (7.2) respectively, if (7.1) holds.

Remark 7.4. (1) LRM for Lévy markets (one dimensional) has been also discussed in Vandaele and Vanmaele [16] without Malliavin calculus. They considered the case where all coefficients in (6.1) are deterministic; and studied LRM for unit-linked life insurance contracts.
(2) Benth et al [4] also concerned a similar issue by using Malliavin calculus. They however studied minimal variance portfolio which is different from LRM, and considered only the case where the underlying asset price process is a martingale.
(3) Yang et al. [17] derived an explicit representation of LRM for a European call option in the Hull and White model by using the Malliavin calculus in Wiener space. They also give a numerical result of it.
(4) Arai and Suzuki [3] derived explicit representations of LRM for one dimensional Lévy markets. They also calculated its concrete expressions for call options, Asian options and lookback options.
(5) Arai et al. [1] illustrate how to compute LRM of call options for exponential Lévy models by using the result of [3] and the fast Fourier transform method.
(6) Arai et al.[2] obtained explicit representations of LRM of call and put options for the Barndorff-Nielsen and Shephard models, which are Ornstein-Uhlenbeck-type stochastic volatility models. They also investigated the Malliavin differentiability of the density of the minimal martingale measure. Moreover, they gave some numerical experiments for LRM strategies.

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