Communications on Stochastic Analysis

Volume 7 | Number 3 Article 7

9-1-2013

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Communications on Stochastic Analysis
Vol. 7, No. 3 (2013) 459-479

Serials Publications
www.serialspublications.com

A BOCHNER-TYPE REPRESENTATION OF POSITIVE DEFINITE MAPPINGS ON THE DUAL OF A COMPACT GROUP

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Abstract. For an arbitrary compact group G with dual space $\Sigma(G)$ a Bochner-type bijection is established between positive definite mappings on $\Sigma(G)$ and central bounded measures on G. This bijection defined by the generalized Fourier transformation is based on the comparative study of three kinds of function algebras: the coefficient algebra, the algebra of convergent Fourier series, and the central Fourier algebra of G.

1. Introduction

Bochner's classical theorem on the characterization of continuous positive definite functions on the real line as Fourier transforms of bounded measures has been generalized to various algebraic-topological structures such as abelian groups, Gelfand pairs and commutative hypergroups, to name only a few examples.

If G is a locally compact abelian group and $G^{\wedge}$ is its dual group, then for any continuous positive definite function ϕ on $G^{\wedge}$ there exists a unique bounded measure τ on G such that ϕ is represented as the Fourier transform of τ.

This representation admits analogues provided for the underlying abstract algebraic-topological structure a dual structure can be defined on which some sort of positive definiteness makes sense.

Within the category of locally compact groups G, successful attempts have lead to establishing a Bochner representation beyond the abelian case, in particular under the assumption that G is a compact Lie group. Even for this class of groups, for which the set $\Sigma(G)$ of irreducible unitary representations serves as the dual object, a new version of positive definiteness has to be invented in order to reach the desired representation. See f.e.[6]. Moreover, there is an axiomatic approach to the Bochner property applying positive definite forms on the coefficient algebra of G in place of positive definite functions on $\Sigma(G)$ ([4]).

In the present paper an application of the central Fourier algebra studied in the unpublished Doctoral dissertation of D.R. Beldin [2] enables us to generalize a Bochner type theorem from connected compact Lie groups to arbitrary compact groups.

Received 2013-8-16; Communicated by D. Applebaum.
2010 *Mathematics Subject Classification.* Primary 22C05; Secondary 43A30, 43A35.
Key words and phrases. Central measures and functions, Fourier transform, positive definite mappings, Bochner representation.

The inspiration for writing up previous thoughts on the subject going back to the author's monographs [4] and [5] is due to D. Applebaum who in his upcoming book [1] points out that the harmonic analysis of compact groups still lacks a precise analogue of Bochner's theorem.

The main result to be presented here states that for every positive definite mapping ϕ on the dual $\Sigma(G)$ of a compact group G there exists a unique central measure μ on G such that its Fourier transform $\hat{\mu}$ equals ϕ.

The layout of our exposition is determined by the interplay among three types of function algebras.

Chapter 2 contains preliminaries from the representation theory of a compact group G. In particular the *coefficient algebra* $F(G)$ of G and the Fourier–Stieltjes transform are introduced.

In Chapter 3, we recall the properties of central measure and functions on G and study the central Fourier mapping. Moreover, we introduce the *algebra* $K(G)$ *of absolutely convergent Fourier series* whose spectrum coincides with G.

This result is an important tool in approaching a similar identification for the *central Fourier algebra* $F^0(G)$ of G which is the subject of Chapter 4. This identification is achieved by extending elements of $\mathrm{spec}(F^0(G))$ for elements in $\mathrm{spec}(K(G))$, where the extension procedure is purely Banach-algebraic, similar to but different from an idea in [3], (34.37). Finally, Chapter 5 is devoted to introducing a suitable notion of positive definite mappings on the dual $\Sigma(G)$ of G and to proving Bochner's theorem along the lines of its classical predecessor, by applying the spectral properties of $F^0(G)$ in place of $L^1(G)$. The necessary arguments are borrowed from [2]. Although it has been attempted to make the paper more easily readable by reproducing well-known facts and referring consequently to the seminal monograph [3], the reader is expected to be familiar with the Gelfand theory for commutative Banach algebras and its application to locally compact abelian groups ([8]).

2. Preliminaries from Representation Theory

For a multiplicatively written locally compact group G with unit element e the space of continuous functions on G will be denoted by $C(G)$. On G there exists a Haar measure ω_G which is unique within a positive factor. ω_G is bounded and hence normalizable to 1 if and only if G is compact. $M(G)$ denotes the Banach space of bounded (Radon) measures on G, and $L^1(G, \omega_G)$ the norm-closed subspace of ω_G-absolutely continuous measures. In fact, $M(G)$ is a Banach $*$-algebra with respect to convolution and conjugation, admitting the Dirac measure ε_e as multiplicative unit.

We denote by $\mathfrak{B}(\mathcal{H})$ the $*$-algebra of bounded linear operators on a Hilbert space $\mathcal{H}$, with I as the identity operator.

From now on we shall assume G to be a compact group.

Let $\Sigma(G)$ and $\Sigma'(G)$ denote the sets of equivalence classes of irreducible and finite-dimensional (continuous, unitary) representations of G respectively. Since G is compact, $\Sigma(G) \subset \Sigma'(G)$. Given $\sigma \in \Sigma'(G)$ the *character* of σ is defined by

$$\chi_\sigma(x) := \chi_{U^{(\sigma)}}(x) := \mathrm{tr}\left(U^{(\sigma)}(x)\right)$$

for all $x \in G$, where $U^{(\sigma)}$ is a representative of the class σ.

It is a basic result of the harmonic analysis of compact groups G that every $\sigma' \in \Sigma'(G)$ admits a decomposition

$$\sigma' = \bigoplus_{\sigma \in \Sigma(G)} M(\sigma, \sigma')\sigma,$$

where

$$M(\sigma, \sigma') := \int_G \chi_\sigma(x^{-1})\chi_{\sigma'}(x)\omega_G(\mathrm{d}x)$$

stands for the *multiplicity* of σ in σ' ([3], (27.30)). Let $\sigma \in \Sigma(G)$. We choose $U^{(\sigma)} \in \sigma$ with representing Hilbert space $\mathcal{H}_\sigma$ having an orthonormal basis $\{h_1^{(\sigma)}, \ldots, h_{d_\sigma}^{(\sigma)}\}$, where $d_\sigma = \dim \sigma := \dim \mathcal{H}_\sigma$. For $i, j = 1, \ldots, d_\sigma$ the functions

$$x \mapsto u_{ij}^{(\sigma)}(x) := \langle U^{(\sigma)}(x)h_j^{(\sigma)}, h_i^{(\sigma)}\rangle$$

are the *coefficient functions* of σ.

The spaces

$$F_\sigma(G) := \left\langle \left\{ u_{ij}^{(\sigma)} : i, j = 1, \ldots, d_\sigma \right\} \right\rangle, \quad (\sigma \in \Sigma(G))$$

and

$$F(G) := \left\langle \bigcup_{\sigma \in \Sigma(G)} F_\sigma(G) \right\rangle$$

are independent of the choice of the representative $U^{(\sigma)}$ of σ.

$F(G)$ is a subalgebra of $C(G)$ called the *coefficient algebra* of G.

A second basic result of the harmonic analysis of compact groups G is the Peter–Weyl theorem which states that $F(G)$ is dense in $C(G)$ with respect to the uniform norm $\|\cdot\|_u$, and that the system

$$\left\{ \sqrt{d_\sigma} u_{ij}^{(\sigma)} : \sigma \in \Sigma(G),\ i, j = 1, \ldots, d_\sigma \right\}$$

is an orthonormal basis of $L^2(G, \omega_G)$.

In order to introduce the Fourier mapping on $M(G)$ we need some preparations on operators on Hilbert spaces. Let $\{\mathcal{H}_i : i \in I\}$ be a family of Hilbert spaces $\mathcal{H}_i$ of dimension $d_i < \infty$ $(i \in I)$. The set

$$\mathcal{E}(I) := \prod_{i \in I} \mathfrak{B}(\mathcal{H}_i)$$

is a $*$-algebra with respect to the familiar operations. In $\mathcal{E}(I)$ one introduces various norms:

For $E := (E_i)_{i \in I} \in \mathcal{E}(I)$ and $(a_i)_{i \in I}$ with $a_i \geq 1$ for all $i \in I$ one defines

$$\|E\|_p := \begin{cases} \left(\sum_{i \in I} a_i \|E_i\|_{\phi_p}^p\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty \\ \sup\{\|E_i\|_{\phi_\infty} : i \in I\} & \text{if } p = \infty \end{cases}$$

and consider the subspaces

$$\mathcal{E}_p(I) := \{E \in \mathcal{E}(I) : \|E\|_p < \infty\} \quad (1 \leq p \leq \infty).$$

Here

$$\phi_p(u_1,\ldots,u_n) := \begin{cases} \left(\sum_{j=1}^n |u_j|^p\right)^{\frac{1}{p}} & \text{if } 1 \le p < \infty \\ \max\{|u_1|,\ldots,|u_n|\} & \text{if } p = \infty \end{cases}$$

and

$$\|X\|_p := \phi_p(x_1,\ldots,x_u),$$

where $x_1,..,x_n$ are the eigenvalues of the positive definite operator $|X| := (XX^*)^{\frac{1}{2}}$. For details see [3], (28.24).

Now we return to the given compact group G and its *dual* $\Sigma(G)$.

For each $\sigma \in \Sigma(G)$ we choose a representative $U^{(\sigma)}$ with corresponding Hilbert space $\mathcal{H}_\sigma$ of dimension d_σ.

Then

$$\mathcal{E}(\Sigma(G)) := \prod_{\sigma\in\Sigma(G)} \mathfrak{B}(\mathcal{H}_\sigma)$$

is a $*$-algebra, and we obtain Banach spaces

$$\mathcal{E}_p(\Sigma(G)) := \left\{E \in \mathcal{E}\big(\Sigma(G)\big) : \|E\|_p < \infty\right\},$$

where

$$\|E\|_p := \begin{cases} \left(\sum_{\sigma\in\Sigma(G)} d_\sigma \left(\|E_\sigma\|_{\phi_p}\right)^p\right)^{\frac{1}{p}} & \text{if } 1 \le p < \infty \\ \sup\{\|E_\sigma\|_{\phi_\infty}\} : \sigma \in \Sigma(G)\ \} & \text{if } p = \infty \end{cases}$$

In fact, $\mathcal{E}_\infty(\Sigma(G))$ becomes a Banach *-algebra. For a measure $\mu \in M(G)$ the *Fourier*(–Stieltjes) *transform*

$$\hat{\mu} : \Sigma(G) \to \mathcal{E}(\Sigma(G))$$

is given by

$$\langle\hat{\mu}(\sigma)h, k\rangle := \int_G \langle\bar{U}^{(\sigma)}(x)h, k\rangle\mu(dx)$$

for all $\sigma \in \Sigma(G), h, k \in \mathcal{H}_\sigma$. If $\mu := f\omega_G$ with $f \in L^1(G,\omega_G)$, then $\hat{\mu}$ yields the Fourier transform of f.

It is shown in [3], (28.36) that the *Fourier mapping*

$$\mathcal{F} : M(G) \to \mathcal{E}_\infty(\Sigma(G))$$

given by

$$\mathcal{F}(\mu) := \hat{\mu}$$

for all $\mu \in M(G)$ is a norm-decreasing *-isomorphism of Banach *-algebras.

3. Central Measures and Functions

As before we keep the assumption that G is a compact group.

A measure $\mu \in M(G)$ is called *central* if it belongs to the center $M^z(G)$ of $M(G)$ which means that $\mu * \nu = \nu * \mu$ whenever $\nu \in M(G)$.

Theorem 3.1. ([3], (28.48)) *For each $\mu \in M(G)$ the following statements are equivalent:*

(i) $\mu \in M^z(G)$.

(ii) $\mu * u_{ij}^{(\sigma)} = u_{ij}^{(\sigma)} * \mu$ *for any system* $\{u_{ij}^{(\sigma)} : i,j = 1,\dots,d_\sigma\}$ *of coefficients of* $\sigma \in \Sigma(G)$.

(iii) $\mu * \varepsilon_a = \varepsilon_a * \mu$ *for all* a *from a dense subset of* G.

(iv) *Given any* $\sigma \in \Sigma(G)$ *there exists an* $a(\tau,\sigma) \in \mathbb{C}$ *such that*

$$\hat{\mu}(\sigma) = a(\mu,\sigma) I_{d_\sigma}.$$

A function $f \in C(G)$ is said to be *central* provided $f\omega_G \in M^z(G)$.

There are equivalences analogous to the ones in Theorem 3.1 for the set $C^z(G)$ of central functions in $C(G)$ in place of $M^z(G)$.

Centrality of measures and functions on G can also be described as follows: For $f \in C(G)$ let

$$f^0(x) := \int_G f(yxy^{-1})\omega_G(dy)$$

whenever $x \in G$. The mapping

$$P\colon C(G) \to C(G)$$

given by

$$P(f) := f^0$$

for all $f \in C(G)$ is continuous.

Given $\mu \in M(G)$ we define $\mu^0 \in M(G)$ by

$$\mu^0(f) := \mu(f^0)$$

for all $f \in C(G)$. Then $\mu \in M^z(G)$ if and only if $\mu = \mu^0$. From the fact that for $f \in C(G)$, $f = f^0$ if and only if $f(x) = f(yxy^{-1})$ holds whenever $x, y \in G$ it follows that P is surjective from $C(G)$ onto $C^z(G)$.

Now we specify the Fourier transform for measures $\mu \in M^z(G)$ and functions $f \in C^z(G)$.

Definition 3.2. For $\mu \in M^z(G)$ the mapping $\hat{\mu}\colon \Sigma(G) \to \mathbb{C}$ given by

$$\dot{\hat{\mu}}(\sigma) := \int_G \chi_\sigma d\mu$$

for all $\sigma \in \Sigma(G)$ is said to be the *central Fourier transform* of μ.

If $f \in C^z(G)$, then $\hat{\mu} = \widehat{f\omega_G}$ defines the central Fourier transform $\hat{f}$ of f.

Theorem 3.3. *The central Fourier mapping*

$$\dot{F} : M^2(G) \to \mathbb{C}^{\Sigma(G)}$$

given by

$$\dot{F}(\mu) := \dot{\hat{\mu}}$$

for all $\mu \in M^z(G)$ *is injective.*

In order to prove this theorem we consider the $*$-subalgebra

$$F^z(G) := F(G) \cup C^z(G)$$

of the coefficient algebra $F(G)$ of G and prepare the proof by showing the following

Lemma 3.4. $F^z(G) = \langle\{\chi_\sigma : \sigma \in \Sigma(G)\}\rangle$

Proof. We first establish the inclusion

$$\langle\{\chi_\sigma : \sigma \in \Sigma(G)\}\rangle \subset F^z(G).$$

In fact, let $\sigma \in \Sigma(G)$. Then χ_σ is continuous. Moreover, χ_σ is central, since for all $x, y \in G$

$$\begin{aligned} \chi_\sigma(y^{-1}xy) &= \operatorname{tr}\left(U^{(\sigma)}(y^{-1}xy)\right) \\ &= \operatorname{tr}\left(U^{(\sigma)}(x)\right) = \chi_\sigma(x). \end{aligned}$$

For the reverse inclusion we observe that

$$F^z(G) = P(F(G)),$$

where $P(f) := f^0$ for all $f \in C(G)$, hence every element of $F^z(G)$ is of the form f^0 for some $f \in F(G)$. The mapping P being linear it remains to be shown that for each coefficient function $u_{ij}^{(\sigma)}$ of σ the function $\left(u_{ij}^{(\sigma)}\right)^0$ is a multiple of χ_σ. But this follows from the subsequent equalities valid for all $x \in G$:

$$\begin{aligned} \left(u_{ij}^{(\sigma)}\right)^0(x) &= \int_G u_{ij}^{(\sigma)}(yxy^{-1})\omega_G(\mathrm{d}y) \\ &= \int_G \sum_{k,l=1}^{d_\sigma} u_{ik}^{(\sigma)}(y)u_{kl}^{(\sigma)}(x)u_{lj}^{(\sigma)}(y^{-1})\omega_G(\mathrm{d}y) \\ &= \sum_{k,l=1}^{d_\sigma} u_{kl}^{(\sigma)}(x)\delta_{kl}\delta_{ij}\frac{1}{d_\sigma} \\ &= \frac{\delta_{ij}}{d_\sigma}\sum_{k,l=1}^{d_\sigma} u_{kl}^{(\sigma)}(x)\delta_{kl} \\ &= \frac{\delta_{ij}}{d_\sigma}\chi_\sigma(x). \end{aligned}$$

□

Proof of Theorem 3.3. Clearly, $M^z(G)$ is a linear space, and $\dot{\mathcal{F}}$ is linear on $M^z(G)$. Therefore it suffices to show that $\ker\dot{\mathcal{F}} = \{0\}$.

Let $\mu \in M^z(G)$ be given with $\dot{\hat{\mu}} = 0$ for all $\sigma \in \Sigma(G)$. Since μ is central, we have to justify that $\mu(f) = 0$ for all $f \in C^z(G)$. By assumption $0 = \dot{\hat{\mu}}(\sigma) = \mu(\chi_\sigma)$ for $\sigma \in \Sigma(G)$. Now we apply the Peter-Weyl theorem in order to obtain

$$\overline{F(G)}^{\|\cdot\|_u} = C(G).$$

Moreover we know from the Lemma that

$$F^z(G) = \left\langle\{\chi_\sigma : \sigma \in \Sigma(G)\}\right\rangle.$$

Since the mapping $P\colon C(G) \to C^z(G)$ introduced above by $P(f) := f^0$ for all $f \in C(G)$ is continuous and surjective, we obtain

$$\overline{F^z(G)}^{\|\cdot\|_u} = C^z(G).$$

From this follows that $\mu(f) = 0$ for all $f \in C^z(G)$. □

A further algebra to be applied in the sequel is that of absolutely convergent Fourier series. We follow the approach in [3], §34.

Given a measure $\mu \in M(G)$ one considers the associated family $A := \{A_\sigma \in \Sigma(G)\}$ of *coefficient operators* defined by

$$\langle A_\sigma h, k\rangle := \int_G \left\langle U^{(\sigma)}(x^{-1})h, k\right\rangle \mu(\mathrm{d}x)$$

for all h, k in the representing Hilbert space $\mathcal{H}_\sigma$ of A_σ. Clearly, $A_\sigma = D_\sigma \hat{\mu}(\sigma)^* D_\sigma$, where D_σ denotes the conjugation operator for $\sigma \in \Sigma(G)$ in the sense of (27.25) of [3]. Moreover, $\|A\|_p = \|\hat{\mu}\|_p$ whenever $1 \le p \le \infty$. The formal expression

$$\sum_{\sigma\in\Sigma(G)} d_\sigma \mathrm{tr}\,(A_\sigma U^{(\sigma)})$$

is called the *Fourier* (-Stieltjes) *series* of μ, in symbols

$$\mu \sim \sum_{\sigma\in\Sigma(G)} d_\sigma \mathrm{tr}\,(A_\sigma U^{(\sigma)}).$$

Analogously one introduces the notion of Fourier series of a function $f \in L^1(G, \omega_G)$. The Fourier series

$$f \sim \sum_{\sigma\in\Sigma(G)} d_\sigma \mathrm{tr}\,(A_\sigma U^{(\sigma)})$$

of $f \in L^1(G, \omega_G)$ is said to be *absolutely convergent* provided

$$\sum_{\sigma\in\Sigma(G)} d_\sigma \|A_\sigma\|_{\phi_1} < \infty.$$

Let $K(G)$ denote the set of functions in $L^1(G, \omega_G)$ which admit on absolutely convergent Fourier series. For $f \in K(G)$ we define

$$\|f\|_{\phi_1} := \|\hat{f}\|_1,$$

where

$$\begin{aligned} \|\hat{f}\|_1 &= \sum_{\sigma\in\Sigma(G)} d_\sigma \|\hat{f}(\sigma)\|_{\phi_1} \\ &= \sum_{\sigma\in\Sigma(G)} d_\sigma \|A_\sigma\|_{\phi_1} < \infty. \end{aligned}$$

$K(G)$ is a linear space, $\mathcal{F}\colon K(G) \to \mathcal{E}_1(\Sigma(G))$ is a norm-preserving linear isomorphism, hence $K(G)$ is a Banach space.

Since any $f \in K(G)$ with

$$f \sim \sum_{\sigma\in\Sigma(G)} d_\sigma \mathrm{tr}\,(A_\sigma U^{(\sigma)})$$

is ω_G-almost everywhere equal to the continuous function

$$\sum_{\sigma\in\Sigma(G)} d_\sigma \mathrm{tr}\,(A_\sigma U^{(\sigma)}),$$

it can be regarded as an element of $C(G)$.

Theorem 3.5. *(i) $K(G)$ is a commutative Banach algebra under pointwise operations, having 1 as its unit.*

(ii) $\mathrm{spec}(K(G)) = D(K(G))$, *i.e. every multiplicative linear functional on $K(G)$ is an evaluation functional of the form*

$$f \mapsto f(a) = \varepsilon_a(f)$$

for a fixed $a \in G$. Furnished with the Gelfand topology the set $\{\varepsilon_a : a \in G\}$ is homeomorphic to G.

A *proof* of this theorem is contained in [3], §34. We quote the essential arguments.

(i) (34.18) Basically it has to be shown that for $f, g \in K(G)$ with convergent Fourier series

$$f = \sum_{m \geq 1} d_m \mathrm{tr}\,(A_m U^{(m)})$$

and

$$g = \sum_{n \geq 1} d_n \mathrm{tr}\,(B_n U^{(n)})$$

respectively the inequalities

$$\begin{aligned} \|fg\|_{\phi_1} &\leq \sum_{m\geq 1}\sum_{n\geq 1} d_m d_n \|\mathrm{tr}\,(A_m U^{(m)})\mathrm{tr}\,(B_n U^{(m)})\|_{\phi_1} \\ &\leq \|f\|_{\phi_1}\|g\|_{\phi_1} \end{aligned}$$

hold, where the crucial estimate is

$$\|\mathrm{tr}\,(A_m U^{(m)})\mathrm{tr}\,(B_n U^{(m)})\|_{\phi_1} \leq \|A_m\|_{\phi_1}\|B_n\|_{\phi_1}$$

valid for all $m, n \geq 1$.

(ii) (34.20) Since $\mathcal{F}$ is a Banach space isomorphism $K(G) \to \mathcal{E}_1(\Sigma(G))$, each bounded linear functional L on $K(G)$ is of the form

$$f \mapsto L(f) := \sum_{\sigma \in \Sigma(G)} d_\sigma \mathrm{tr}\,(\hat{f}(\sigma) F_\sigma),$$

where

$$F := \{F_\sigma : \sigma \in \Sigma(G)\} \in \mathcal{E}_\infty(\Sigma(G))$$

and

$$\sup\{\|F_\sigma\|_{\phi_\infty} : \sigma \in \Sigma(G)\} = 1.$$

From the multiplicativity of L one deduces that L is an evaluation functional on $F(G)$. But $F(G)$ is dense in $K(G)$ with respect to the norm $\|\cdot\|_{\phi_1}$ and L is continuous. Consequently L is an evaluation functional on $K(G)$.

So far one knows that $a \mapsto \phi(a) := \varepsilon_a$ is a one-to-one mapping from G into $\mathrm{spec}(K(G))$. The inclusion $K(G) \subset C(G)$ and the definition of the Gelfand topology yield the continuity of ϕ. As $\mathrm{spec}(K(G))$ is compact (as the structure space of $K(G)$), ϕ is in fact a homeomorphism.

4. The Central Fourier Algebra

Next to the algebras $F(G)$ and $K(G)$ associated with the given compact group G we shall place special emphasis on the *central Fourier algebra* $F^0(G)$ of G consisting of all functions $f \in C^z(G)$ that admit convergent Fourier series in the sense that

$$f = \sum_{\sigma \in \Sigma(G)} \dot{\hat{f}}(\sigma^*)\chi_\sigma$$

with

$$\|f\|^0 := \sum_{\sigma \in \Sigma(G)} d_\sigma |\dot{\hat{f}}(\sigma)| < \infty.$$

In analogy to the respective result for $K(G)$ we have the following

Theorem 4.1. *$F^0(G)$ is a commutative Banach *-algebra with respect to the usual pointwise operations, complex conjugation as involution, and $\|\cdot\|^0$ as norm.*

Proof. In order to show that $F^0(G)$ is a commutative normed *-algebra we restrict ourselves to proving the crucial inequality

$$\|fg\|^0 \leq \|f\|^0 \|g\|^0$$

valid for all $f, g \in F^0(G)$.

At first we note that for $f, g \in F^0(G), \sigma \in \Sigma(G)$

$$\dot{\widehat{fg}}(\sigma) = \sum_{\sigma_1, \sigma_2 \in \Sigma(G)} \dot{\hat{f}}(\sigma_1^*)\dot{\hat{g}}(\sigma_2^*) M(\sigma, \sigma_1^* \otimes \sigma_2^*).$$

In fact,

$$\begin{aligned} fg &= \Big(\sum_{\sigma_1 \in \Sigma(G)} \dot{\hat{f}}(\sigma_1^*)\chi_{\sigma_1}\Big)\Big(\sum_{\sigma_2 \in \Sigma(G)} \dot{\hat{g}}(\sigma_2^*)\chi_{\sigma_2}\Big) \\ &= \sum_{\sigma_1, \sigma_2 \in \Sigma(G)} \dot{\hat{f}}(\sigma_1^*)\dot{\hat{g}}(\sigma_2^*)\chi_{\sigma_1 \otimes \sigma_2}, \end{aligned}$$

hence

$$\dot{\widehat{fg}}(\sigma) = \sum_{\sigma_1 \sigma_2 \in \Sigma(G)} \dot{\hat{f}}(\sigma_1^*)\dot{\hat{g}}(\sigma_2^*) \dot{\widehat{\chi_{\sigma_1 \otimes \sigma_2}}}(\sigma),$$

where

$$\begin{aligned} \dot{\widehat{\chi_{\sigma_1 \otimes \sigma_2}}}(\sigma) &= \int_G \chi_\sigma(x)\chi_{\sigma_1 \otimes \sigma_2}(x)\omega_G(\mathrm{d}x) \\ &= \int_G \chi_{\sigma^*}(x^{-1})\chi_{\sigma_1 \otimes \sigma_2}(x)\omega_G(\mathrm{d}x) \\ &= M(\sigma^*, \sigma_1 \otimes \sigma_2) \\ &= M(\sigma, (\sigma_1 \otimes \sigma_2)^*) \\ &= M(\sigma, \sigma_1^* \otimes \sigma_2^*). \end{aligned}$$

Now we compute the $\|\cdot\|^0$-norm of the product fg:

$$\begin{aligned}
\|fg\|^0 &= \sum_{\sigma\in\Sigma(G)} \left|\dot{\widehat{fg}}\right| d_\sigma \\
&= \sum_{\sigma\in\Sigma(G)} \Big| \sum_{\sigma_1,\sigma_2\in\Sigma(G)} \dot{\hat{f}}(\sigma_1^*)\dot{\hat{g}}(\sigma_2^*)M(\sigma,\sigma_1^*\otimes\sigma_2^*)\Big| d_\sigma \\
&\le \sum_{\sigma\in\Sigma(G)} \sum_{\sigma_1,\sigma_2\in\Sigma(G)} \big|\dot{\hat{f}}(\sigma_1^*)\big|\big|\dot{\hat{g}}(\sigma_2^*)\big| M(\sigma,\sigma_1^*\otimes\sigma_2^*) d_\sigma \\
&= \sum_{\sigma_1,\sigma_2\in\Sigma(G)} \big|\dot{\hat{f}}(\sigma_1^*)\big|\big|\dot{\hat{g}}(\sigma_2^*)\big| d_{\sigma_1^*\otimes\sigma_2^*} \\
&= \sum_{\sigma_1,\sigma_2\in\Sigma(G)} \big|\dot{\hat{f}}(\sigma_1^*)\big|\big|\dot{\hat{g}}(\sigma_2^*)\big| d_{\sigma_1^*} d_{\sigma_2^*} \\
&= \|f\|^0\|g\|^0.
\end{aligned}$$

Finally we have to show that $F^0(G)$ is in fact a Banach*-algebra. For this purpose we need to provide an embedding of $F^0(G)$ into a Banach*-algebra $L^1(\Sigma(G))$ of integrable functions on the discrete dual $\Sigma(G)$ of G. More precisely, on the power set $\mathcal{P}(\Sigma(G))$ of $\Sigma(G)$ one defines a mapping d into $\bar{\mathbb{R}}_+$ by

$$\mathrm{d}(\Sigma_1) := \begin{cases} \sum_{\sigma\in\Sigma_1} d_\sigma & \text{if } \Sigma_1 \text{ is finite} \\ \infty & \text{otherwise} \end{cases}$$

Then $(\Sigma(G), \mathcal{P}(\Sigma(G)), \mathrm{d})$ is a positive measure space. Let $L(\Sigma(G))$ denote the set of complex-valued functions on $\Sigma(G)$ with finite support, and let $L^1(\Sigma(G))$ be the set of d-integrable functions on $\Sigma(G)$. Clearly, $L(\Sigma(G))$ is a dense subset of $L^1(\Sigma(G))$ with respect to the norm of $L^1(\Sigma(G))$. In $L^1(\Sigma(G))$ we introduce a multiplication by

$$(\phi\times\psi)(\sigma) := \sum_{\sigma_1,\sigma_2\in\Sigma(G)} \phi(\sigma_1)\psi(\sigma_2)M(\sigma,\sigma_1\otimes\sigma_2)$$

and an involution by

$$\phi^*(\sigma) := \overline{\phi(\sigma^*)}$$

for all $\phi,\psi\in L^1(\Sigma(G)), \sigma\in\Sigma(G)$. Then $L^1(\Sigma(G))$ becomes a Banach*-algebra, and the mapping $f\mapsto\dot{\hat{f}}$ from $F^0(G)$ into $L^1(\Sigma(G))$ is an isometric *-isomorphism between algebras. As a consequence we see that $F^0(G)$ is a Banach *-algebra. □

Theorem 4.2. *$F^0(G)$ coincides with the Banach *-algebra $K^z(G)$ of central functions in $K(G)$.*

Proof. 1. $F^0(G)\subset K^z(G)$.

In fact, let $f \in F^0(G)$. Then $f \in C(G)$, hence $f \epsilon L^1(G, \omega_G)$. Applying Theorem 3.1 we obtain

$$\begin{aligned} \|f\|_{\phi_1} &= \sum_{\sigma \in \Sigma(G)} d_\sigma \|\dot{\hat{f}}(\sigma)\|_{\phi_1} \\ &= \sum_{\sigma \in \Sigma(G)} d_\sigma \|a(f,\sigma) I_{d_\sigma}\|_{\phi_1} \\ &= \sum_{\sigma \in \Sigma(G)} d_\sigma^2 |a(f,\sigma)| \\ &= \sum_{\sigma \in \Sigma(G)} d_\sigma^2 \frac{|\dot{\hat{f}}(\sigma)|}{d_\sigma} = \|f\|^0 < \infty. \end{aligned}$$

Together with the fact that the functions of $F^0(G)$ are central this implies the stated inclusion.

2. $K^z(G) \subset F^0(G)$.

Let $f \in K^z(G)$. Since f is central, Theorem 3.1 yields

$$\hat{f}(\sigma) = a(f,\sigma) I_{d_\sigma}$$

for some $a(f,\sigma) \in \mathbb{C}$, all $\sigma \in \Sigma(G)$. Now consider the function

$$g := \sum_{\sigma \in \Sigma(G)} a(f,\sigma) d_\sigma \chi_\sigma$$

on G. Applying the formula

$$\hat{\chi}_\sigma(\tau) = \frac{\delta_{\sigma\tau}}{d_\tau} I_{d_\sigma}$$

valid for all $\tau \in \Sigma(G)$, derived from the Peter-Weyl theorem, one sees that $\hat{g} = \hat{f}$, hence by the injectivity of the Fourier transform that $g = f$. But now $\dot{\hat{f}}(\sigma^*) = a(f,\sigma) d_\sigma$ for all $\sigma \in \Sigma(G)$. Consequently

$$\begin{aligned} \infty > \|f\|_{\phi_1} &= \sum_{\sigma \in \Sigma(G)} d_\sigma \|\hat{f}(\sigma)\|_{\phi_1} \\ &= \sum_{\sigma \in \Sigma(G)} d_\sigma |d_\sigma a(f,\sigma)| \\ &= \sum_{\sigma \in \Sigma(G)} d_\sigma |\dot{\hat{f}}(\sigma^*)| \\ &= \sum_{\sigma \in \Sigma(G)} d_\sigma |\dot{\hat{f}}(\sigma)| = \|f\|^0 \end{aligned}$$

It remains to show the continuity of f.

We know from the discussion in Chapter 3 that for $\{A_\sigma : \sigma \in \Sigma(G)\} \in \mathcal{E}_1(\Sigma(G))$ the function

$$x \mapsto h(x) := \sum_{\sigma \in \Sigma(G)} d_\sigma \operatorname{tr}(A_\sigma U^{(\sigma)}(x))$$

is continuous on G. Since $f \in K^z(G)$,

$$f = \sum_{\sigma \in \Sigma(G)} a(f,\sigma) d_\sigma \chi_\sigma$$

and

$$\hat{f}(\sigma) = a(f,\sigma) I_{d_\sigma} = \hat{f}(\sigma)^t$$

or

$$A_\sigma = \hat{f}(\sigma)^t = \hat{f}(\sigma)$$

for all $\sigma \in \Sigma(G)$, hence $\{\hat{f}(\sigma) : \sigma \in \Sigma(G)\} \in \mathcal{E}_1(\Sigma(G))$. Moreover, for all $x \in G$

$$\begin{aligned} h(x) &= \sum_{\sigma \in \Sigma(G)} d_\sigma \mathrm{tr}\,(A_\sigma U^{(\sigma)}(x)) \\ &= \sum_{\sigma \in \Sigma(G)} d_\sigma \mathrm{tr}\,(a(f,\sigma) I_{d_\sigma} U^{(\sigma)}(x)) \\ &= \sum_{\sigma \in \Sigma(G)} d_\sigma a(f,\sigma) \mathrm{tr}\,(U^{(\sigma)}(x)) \\ &= \sum_{\sigma \in \Sigma(G)} d_\sigma a(f,\sigma) \chi_\sigma(x). \end{aligned}$$

This completes the proof. □

Theorem 4.3. *G acts on $K(G)$ as a compact group of continuous automorphisms of G.*

Proof. For each $x \in G$ consider the mapping $i_x : G \to G$ given by

$$i_x(y) := xyx^{-1}$$

for all $y \in G$. Moreover, let the mapping $i \in G \to K(G)$ be defined by

$$i(x)(f) := f \circ i_x$$

for all $f \in K(G), x \in G$. Then $i(x)(f) \in K(G)$.

In fact, we have the two equalities

$$i(x)(f) = f \circ i_x \in L^1(G, \omega_G)$$

and

$$\begin{aligned} \|i(x)(f)\|_{\phi_1} &= \sum_{\sigma \in \Sigma(G)} d_\sigma \|\widehat{i(x)(f)}(\sigma)\|_{\phi_1} \\ &= \sum_{\sigma \in \Sigma(G)} d_\sigma \|\hat{f}(\sigma)\|_{\phi_1} \\ &= \|f\|_{\phi_1} < \infty. \end{aligned}$$

In order to see the second equality we compute for all $\sigma \in \Sigma(G), h, k \in \mathcal{H}_\sigma$

$$\begin{aligned}\left\langle \widehat{i(x)(f)}(\sigma)h, k\right\rangle &= \int_G \langle \bar{U}^{(\sigma)}(y)h, k\rangle f(xyx^{-1})\omega_G(\mathrm{d}y) \\ &= \int_G \langle \bar{U}^{(\sigma)}(x^{-1}yx)h, k\rangle f(y)\omega_G(\mathrm{d}y) \\ &= \int_G \langle \bar{U}^{(\sigma)}(x^{-1})\bar{U}^{(\sigma)}(y)\bar{U}^{(\sigma)}(x)h, k\rangle f(y)\omega_G(\mathrm{d}y) \\ &= \int_G \langle \bar{V}_x(y)h, k\rangle f(y)\omega_G(\mathrm{d}y) \\ &= \langle \widehat{f}^{(U)}(\sigma)h, k\rangle,\end{aligned}$$

where $\widehat{f}^{(U)}$ denotes the Fourier transform of f with respect to $V_x \in \sigma$. This implies $\widehat{i(x)(f)}(\sigma) = \widehat{f}^{(U)}(\sigma)$, hence

$$\|\widehat{i(x)(f)}(\sigma)\|_{\phi_1} = \|\widehat{f}^{(U)}(\sigma)\|_{\phi_1} = \|\hat{f}(\sigma)\|_{\phi_1}.$$

Next we observe that for every $x \in G$ the mapping $i(x)\colon K(G) \to K(G)$ is an automorphism of $K(G)$. Obviously, $i(x)$ is linear and multiplicative, and since $\|i(x)(f)\|_{\phi_1} = \|f\|_{\phi_1}$ for all $f \in K(G), i(x)$ is continuous. Finally, the mapping $i\colon G \to \mathrm{Aut}(K(G))$ is an anti-automorphism of groups. In fact, for $x, y, z \in G$ and $f \in K(G)$ we obtain

$$\begin{aligned}(i(x) \circ i(y))(f)(z) &= (f \circ i_{xy})(z) \\ &= f(i_{xy})(z) \\ &= f(xyzy^{-1}x^{-1}) \\ &= (f \circ i_x)(yzy^{-1}) \\ &= (i(x)(f) \circ i_y)(z) \\ &= (i(y) \circ i(y)(f)(z).\end{aligned}$$

□

Theorem 4.4. $\mathrm{spec}(F^0(G) = D(F^0(G))$.

Proof. In view of Theorem 4.3

$$F^0(G) = \{f \in K(G) : f = i(x)(f) \text{ for all } x \in G\}$$

is a closed subalgebra of $K(G)$. Let L be a non-vanishing multiplicative linear functional of $F^0(G)$. From [7], Theorem 1.3.3 we infer that there exists a multiplicative linear extension L' of L on $K(G)$. But by Theorem 3.3 $\mathrm{spec}(K(G)) = D(K(G))$, hence there exists an $a \in G$ such that

$$L'(f) = f(a) = \varepsilon_a(f)$$

for all $f \in F^0(G)$. This shows the inclusion $\mathrm{spec}(F^0(G)) \subset D(F^0(G))$. The other inclusion is trivial. □

5. Bochner's Theorem

For the given compact group G we abbreviate by $\Sigma := \Sigma(G)$ and $\Sigma' := \Sigma'(G)$ the sets of equivalence classes of irreducible and finite dimensional (continuous unitary) representations of G respectively.

Definition 5.1. A mapping $\phi\colon \Sigma \to \mathbb{C}$ is called *positive definite* (on Σ) if for every $N \geq 1$, every sequence $\{\sigma_1, \ldots, \sigma_N\}$ in Σ and every sequence $\{c_1, \ldots c_N\}$ in $\mathbb{C}$

$$\sum_{n,m=1}^{N} c_n \bar{c}_m \sum_{\sigma \in \Sigma} M(\sigma, \sigma_n \otimes \sigma_m^*)\phi(\sigma) \geq 0.$$

We denote by $P(\Sigma)$ the set of all positive definite mappings on Σ.

Remark 5.2. Any mapping $\phi : \Sigma \to \mathbb{C}$ can be extended to a mapping $\phi' : \Sigma' \to \mathbb{C}$ by putting

$$\phi'(\sigma') := \sum_{\sigma \in \Sigma} M(\sigma, \sigma')\phi(\sigma)$$

for all $\sigma' \in \Sigma'$. Consequently, $\phi : \Sigma \to \mathbb{C}$ is positive definite if and only if for all $N \geq 1$, sequences $\{\sigma_1, \ldots, \sigma_N\}$ in Σ and $\{c_1, \ldots, c_N\}$ in $\mathbb{C}$

$$\sum_{n,m=1}^{N} c_n \bar{c}_m \phi'(\sigma_n \otimes \sigma_m^*) \geq 0.$$

Special Case 5.3 If G is abelian, the above definition of positive definiteness coincides with the usual one. Indeed, in this case $\Sigma = G^*$, i.e. representations $\sigma \in \Sigma$ are characters $\chi_\sigma \in G^*$, and $\sigma^* = \chi_\sigma^{-1}$ as well as $\sigma \otimes \sigma' = \chi_\sigma \chi_{\sigma'}$ whenever $\sigma, \sigma' \in \Sigma$. Given $\phi \in P(\Sigma)$ we obtain with the choice $N \geq 1$, $\{\sigma_1, \ldots, \sigma_N\}$ in Σ and $\{c_1, \ldots c_N\}$ in $\mathbb{C}$

$$\begin{aligned}
&\sum_{n,m=1}^{N} c_n \bar{c}_m \sum_{\sigma \in \Sigma} \left(\int_G \overline{\chi_\sigma(x)} \chi_{\sigma_n \otimes \sigma_m^*}(x) \omega_G(\mathrm{d}x) \right) \phi(\sigma) \\
&= \sum_{n,m=1}^{N} c_n \bar{c}_m \sum_{\sigma \in \Sigma} \delta_{\sigma, \sigma_n \otimes \sigma_m^*} \phi(\sigma) \\
&= \sum_{n,,m=1}^{N} c_n \bar{c}_m \phi(\sigma_n \otimes \sigma_m^*) \\
&= \sum_{n,m=1}^{N} c_n \bar{c}_m \phi(\chi_n \chi_m^{-1}).
\end{aligned}$$

Properties 5.4 of $\phi \in P(\Sigma)$.
5.4.1 $\phi(I) \geq 0$, where I denotes the trivial one-dimensional representation in Σ.
5.4.2 $\phi(\sigma^*) = \overline{\phi(\sigma)}$ for all $\sigma \in \Sigma$.

Proofs. 5.4.1 Choosing $N = 1$ and $c_1 = 1$ we obtain

$$\sum_{\sigma \in \Sigma} M(\sigma, \sigma_1 \otimes \sigma_1^*)\, \phi(\sigma) \geq 0$$

for all $\sigma_1 \in \Sigma$, in particular for $\sigma_1 = I$

$$\phi(I) = \sum_{\sigma \in \Sigma} M(\sigma, I \otimes I^*)\, \phi(\sigma) \geq 0.$$

5.4.2 With the choices $N = 2, c_1 = 1, c_2 = c, \sigma_1 = \sigma_0$ and $\sigma_2 = I$ we compute

$$\begin{aligned} &\sum_{\sigma \in \Sigma} M(\sigma, \sigma_0 \otimes \sigma_0^*)\, \phi(\sigma) + \bar{c} \sum_{\sigma \in \Sigma} M(\sigma, \sigma_0 \otimes I^*)\, \phi(\sigma) \\ &\quad + c \sum_{\sigma \in \Sigma} M(\sigma, I \otimes \sigma_0^*)\, \phi(\sigma) + |c|^2 \sum_{\sigma \in \Sigma} M(\sigma, I \otimes I^*)\, \phi(\sigma) \\ &= \sum_{\sigma \in \Sigma} M(\sigma, \sigma_0 \otimes \sigma_0^*)\, \phi(\sigma) + \bar{c}\phi(\sigma_0) + c\phi(\sigma_0^*) + |c|^2 \phi(I) \geq 0. \end{aligned}$$

By 5.4.1 the first and the last term of the sum is real and ≥ 0, and hence the sum $\bar{c}\phi(\sigma_0) + c\phi(\sigma_0^*)$ is real. Since this property holds for all $c \in \mathbb{C}$, we choose $c = 1$ and $c = i$ to obtain $\phi(\sigma_0) + \phi(\sigma_0^*) \in \mathbb{R}$ as well as $i\phi(\sigma_0^*) - i\phi(\sigma_0) \in \mathbb{R}$. But then $\overline{\phi(\sigma_0)} = \phi(\sigma_0^*)$ which was to be shown. □

In the following discussion we shall make use of two more symbols: $M_+^z(G)$ for the set of non-negative measure in $M^z(G)$ and $P^b(\Sigma)$ for the set of mappings $\phi \in P(\Sigma)$ that are *bounded* in the sense that there exists a constant $c_\phi \geq 0$ such that

$$|\phi(\sigma)| \leq c_\phi d_\sigma$$

for all $\sigma \in \Sigma$.

Theorem 5.5. *(Bochner) The restriction $\dot{\mathcal{F}}_1$ of the central Fourier mapping $\dot{\mathcal{F}}$ to $M_+^z(G)$ provides a bijection from $M_+^z(G)$ onto $P^b(\Sigma)$.*

Proof. 1. By Theorem 3.3 $\dot{\mathcal{F}}_1$ is injective.

2. We show that for every $\mu \in M^z(G)$ the central Fourier transform $\dot{\hat{\mu}} = \dot{\mathcal{F}}_1(\mu)$ of μ belongs to $P^b(\Sigma)$.

2.1 Let $N \geq 1, \{\sigma_1, \ldots, \sigma_N\} \subset \Sigma$ and $\{c_1, \ldots, c_N\} \subset \mathbb{C}$.

By Remark 5.2 we may consider the extension $\dot{\hat{\mu}}'$ of $\dot{\hat{\mu}}$ to Σ' which is of the form

$$\dot{\hat{\mu}}'(\sigma') = \int_G \chi_{\sigma'} \mathrm{d}\mu$$

for all $\sigma' \in \Sigma'$. In fact, let σ' admit the decomposition

$$\sigma' = \bigoplus_{\sigma \in \Sigma} M(\sigma_1 \sigma)\sigma.$$

Then
$$\chi_{\sigma'} = \sum_{\sigma\in\Sigma} M(\sigma,\sigma')\chi_\sigma,$$
hence
$$\begin{aligned}\dot{\hat{\mu}}'(\sigma') &= \sum_{\sigma\in\Sigma} M(\sigma,\sigma')\dot{\hat{\mu}}(\sigma)\\ &= \sum_{\sigma\in\Sigma} M(\sigma,\sigma')\int_G \chi_\sigma \,\mathrm{d}\mu\\ &= \int_G \Big(\sum_{\sigma\in\Sigma} M(\sigma,\sigma')\chi_\sigma\Big)\,\mathrm{d}\mu\\ &= \int_G \chi_{\sigma'}\,\mathrm{d}\mu.\end{aligned}$$

Now,
$$\begin{aligned}&\sum_{n,m=1}^{N} c_n\bar{c}_m \sum_{\sigma\in\Sigma} M(\sigma,\sigma_n\otimes\sigma_m^*)\dot{\hat{\mu}}(\sigma)\\ &= \sum_{n,m=1}^{N} c_n\bar{c}_m \dot{\hat{\mu}}'(\sigma_n\otimes\sigma_m^*)\\ &= \sum_{m,n=1}^{N} c_n\bar{c}_m \int_G \chi_{\sigma_n\otimes\bar{\chi}_{\sigma_m}}\,\mathrm{d}\mu\\ &= \sum_{m,n=1}^{N} c_n\bar{c}_m \int_G \chi_{\sigma_n}\bar{\chi}_{\sigma_m}\,\mathrm{d}\mu\\ &= \int \Big|\sum_{n=1}^{N} c_n\chi_{\sigma_n}\Big|^2 \mathrm{d}\mu \geq 0.\end{aligned}$$

This shows that $\dot{\hat{\mu}} \in P(\Sigma)$.

2.2 $\dot{\hat{\mu}}$ is bounded and therefore $\dot{\hat{\mu}} \in P^b(\Sigma)$.

In fact, the bound $c_{\dot{\hat{\mu}}} := \|\mu\|$ fulfills the requirements, since $c_{\dot{\hat{\mu}}} \geq 0$ and
$$\big|\dot{\hat{\mu}}(\sigma)\big| = \Big|\int_G \chi_\sigma \mathrm{d}\mu\Big| \leq \int_G |\chi_\sigma|\,\mathrm{d}\mu \leq \int_G d_\sigma \mathrm{d}\mu = d_\sigma\|\mu\|$$
whenever $\sigma\in\Sigma$.

3. We show that given $\phi \in P^b(\Sigma)$ there exists a measure $\mu \in M_+^z(G)$ such that $\phi = \dot{\hat{\mu}} = \dot{\mathcal{F}}_1(\tau)$.

3.1 Let $\phi \in P^b(\Sigma)$ be such that $|\phi(\sigma)| \leq c_\phi d_\sigma$ for all $\sigma\in\Sigma$. For every $f \in F^0(G)$ we introduce
$$T_\phi(f) := \sum_{\sigma\in\Sigma} \phi(\sigma)\dot{\hat{f}}(\sigma).$$

Then T_ϕ is a well-defined linear functional on $F^0(G)$ which is $\|\cdot\|^0$-continuous and satisfies $\|T_\phi\| \leq c_\phi$.

3.1.1 From the inequalities

$$\begin{aligned} |T_\phi(f)| &= \Big|\sum_{\sigma\in\Sigma} \phi(\sigma)\dot{\hat{f}}(\sigma)\Big| \\ &\leq \sum_{\sigma\in\Sigma} |\phi(\sigma)||\dot{\hat{f}}(\sigma)| \\ &\leq \sum_{\sigma\in\Sigma} c_\phi d_\sigma |\dot{\hat{f}}(\sigma)| \\ &= c_\phi \sum_{\sigma\in\Sigma} |\dot{\hat{f}}(\sigma)| d_\sigma \\ &= c_\phi \|f\|^0 < \infty \end{aligned}$$

valid for all $f \in F^0(G)$, follows the well-definedness of T_ϕ.

3.1.2 The linearity of T_ϕ is clear.

3.1.3 The $\|\cdot\|^0$-continuity of T_ϕ has already been shown in 3.1.1. It implies that $\|T_\phi\| \leq c_\phi$.

3.2 Now we introduce a mapping

$$[\cdot\,,\,\cdot] := F^0(G) \times F^0(G) \to \mathbb{C}$$

by

$$[f,g] := T_\phi(f\bar{g})$$

for all $f, g \in F^0(G)$.

Obviously

3.2.1 $f \mapsto [f,g]$ is linear for each $g \in F^0(G)$.

3.2.2 For all $f, g \in F^0(G)$ we have

$$[f,g] = \overline{[g,f]},$$

since

$$\begin{aligned} [f,g] &= T_\phi(f\bar{g}) \\ &= T_\phi(\overline{g\bar{f}}) \\ &= \sum_{\sigma\in\Sigma} \overline{\phi(\sigma^*)}\, \overline{\dot{\widehat{g\bar{f}}}}(\sigma^*) \\ &= \sum_{\sigma\in\Sigma} \phi(\sigma) \overline{\dot{\widehat{g\bar{f}}}}(\sigma^*) \\ &= \overline{T_\phi(g\bar{f})} = \overline{[g,f]}. \end{aligned}$$

Here Property 5.4.2 of $P(\Sigma)$ has been applied.

3.2.3 For all $f \in F^0(G)$ we have $[f, f] \geq 0$

Indeed,

$$\begin{aligned} [f,f] &= T_\phi(f\bar{f}) \\ &= \sum_{\sigma\in\Sigma} \phi(\sigma)\widehat{f\bar{f}}(\sigma) \\ &= \sum_{\sigma\in\Sigma} \phi(\sigma) \sum_{\sigma_1,\sigma_2\in\Sigma} \dot{\hat{f}}(\sigma_1^*)\dot{\hat{f}}(\sigma_2^*)M(\sigma,\sigma_1^*\otimes\sigma_2^*) \\ &= \sum_{\sigma\in\Sigma} \phi(\sigma) \sum_{\sigma_1,\sigma_2\in\Sigma} \dot{\hat{f}}(\sigma_1^*)\overline{\dot{\hat{f}}(\sigma_2)}M(\sigma,\sigma_1^*\otimes\sigma_2^*) \\ &= \sum_{\sigma_1,\sigma_2\in\Sigma} \dot{\hat{f}}(\sigma_1)\overline{\dot{\hat{f}}(\sigma_2)} \sum_{\sigma\in\Sigma} M(\sigma,\sigma_1\otimes\sigma_2^*)\phi(\sigma) \geq 0, \end{aligned}$$

the latter inequality resulting from the positive definiteness of ϕ.

3.2.4 From 3.2.1 and 3.2.4 follows the Schwarz inequality

$$|[f,g]|^2 \leq [f,f][g,g]$$

for all $f, g \in F^0(G)$, which implies

$$|T_\phi(f)|^2 \leq \phi(I)T_\phi(f\bar{f}).$$

This is seen as follows: $1 \in F^0(G)$, and

$$\begin{aligned} [1,1] = T_\phi(1\bar{1}) &= T_\phi(1) \\ &= \sum_{\sigma\in\Sigma} \phi(\sigma)\dot{\hat{1}}(\sigma) \\ &= \phi(I)\dot{\hat{1}}(\sigma) = \phi(I). \end{aligned}$$

Consequently,

$$\begin{aligned} |T_\phi(f)|^2 = |T_\phi(f\bar{1})|^2 &= |[f,1]|^2 \\ &\leq [f,f][1,1] = \phi(I)T_\phi(f\bar{f}). \end{aligned}$$

In order to obtain

3.3 the continuity of T_ϕ in the sup-norm $\|\cdot\|_u$ one applies the inequality

3.3.1 $|T_\phi(f)|^2 \leq (\phi(I))^{1+\frac{1}{2}+\cdots+\frac{1}{2^n}}(T_\phi(h^{2^n})^{\frac{1}{2^n}}$
valid for all $f \in F^0(G)$, where $h := f\bar{f}, h^n := hh^{n-1}$ and h^n is a real-valued function for all $n \geq 1$.

The inequality is easily shown by induction.

3.3.2 Since $F^0(G)$ is a *-algebra, $h^{2^n} \in F^0(G)$ for all $n \geq 1$, and from the $\|\cdot\|^0$-continuity of T_ϕ follows

$$0 \leq T_\phi(h^{2^n}) \leq c_\phi(\|h^{2^n}\|^0)^{\frac{1}{2^n}}$$

or

$$\|T_\phi(f)\|^2 \leq \phi(I)^{1+\frac{1}{2}+\cdots+\frac{1}{2^n}} c_\phi^{\frac{1}{2^n}} (\|h^{2^n}\|^0)^{\frac{1}{2^n}}$$

for all $f \in F^0(G)$.

On the other hand, the form of the spectral radius

$$\begin{aligned}\rho(h) &= \lim_{n\to\infty} \left(\|h^n\|^0\right)^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \left(\|h^{2^n}\|^0\right)^{\frac{1}{2^n}} = \|\tilde{h}\|_{\sup}\end{aligned}$$

of h in terms of the Gelfand transform $\tilde{h}$ of h, considered as an element of $C(\mathrm{spec}(F^0(G)))$ with sup-norm $\|\cdot\|_{\sup}$ yields

$$\begin{aligned}|T_\phi(f)|^2 &\leq \lim_{n\to\infty} \left((\phi(I))^{1+\frac{1}{2}+\cdots+\frac{1}{2^n}} c_\phi^{\frac{1}{2^n}} \left(\|h^{2^n}\|^0\right)^{\frac{1}{2^n}} \right) \\ &= \phi(I)^2 \|\tilde{h}\|_{\sup} \\ &= \phi(I)^2 \|\widetilde{f\bar{f}}\|_{\sup} \\ &= \phi(I)^2 \|\tilde{f}\bar{\tilde{f}}\|_{\sup} \\ &\leq \phi(I)^2 \|\tilde{f}\|_{\sup}^2\end{aligned}$$

for all $f \in F^0(G)$.

To finish the proof of 3.3 it remains to verify that

3.3.2 $\|\tilde{f}\|_{\sup} = \|f\|_u$.

This equality follows from Theorem 4.4, since

$$\begin{aligned}\|\tilde{f}\|_{\sup} &= \sup\left\{|f(L)| : L \in \mathrm{spec}(F^0(G))\right\} \\ &= \sup\left\{|L(f)| : L \in \mathrm{spec}(F^0(G))\right\} \\ &= \sup\left\{|\varepsilon_x(f) : x \in G\right\} \\ &= \sup\left\{|f(x)| : x \in G\right\} = \|f\|_u.\end{aligned}$$

3.4 Since $F^z(G) \subset F^0(G)$, the Peter-Weyl theorem yields the $\|\cdot\|_u$-density of $F^0(G)$ in $C^z(G)$. Consequently there exists a linear, $\|\cdot\|_u$-continuous extension T'_ϕ of T_ϕ to $C^z(G)$. Moreover

3.4.1 T_ϕ is a positive operator.

Indeed, for all $f \in F^0(G)$ we have $T_\phi(f\bar{f}) \geq 0$ by 3.2.3. Since $F^0(G)$ is $\|\cdot\|_u$-dense in $C^z(G)$, for an arbitrary $f \in C^z(G)$, there exists a sequence $(f_n)_{n\geq 1}$ in $F^0(G)$ which $\|\cdot\|_u$-converges to f. But then $(f_n, \bar{f}_n)_{n\geq 1} \|\cdot\|_u$ converges to $f\bar{f}$, and the $\|\cdot\|_u$-continuity of T'_ϕ on $C^z(G)$ implies that

$$T'_\phi(f_n\bar{f}_n) \to T_\phi(f\bar{f})$$

as $n \to \infty$. From $T'_\phi(f_n\bar{f}_n) = T_\phi(f_n\bar{f}_n) \geq 0$ follows $T'_\phi(f\bar{f}) \geq 0$.

Now let f belong to the cone $C^z_+(G)$ of nonnegative functions in $C^z(G)$ and let it be of the form $f = gg = g\bar{g}$ with $g \in C^z_+(G)$. Then $T'_\phi(f) \geq 0$.

3.4.2 Finally, the mapping $\mu : C(G) \to \mathbb{C}$ defined by

$$\mu(f) := T'_\phi(f)$$

for all $f \in C(G)$ is a positive, hence $\| \cdot \|_u$-continuous linear functional on $C(G)$ which by the Riesz theorem is represented by a measure $\mu \in M^z_+(G)$. The measure $\bar{\mu}$ fulfills the requirements of the theorem: For any $\sigma \in \Sigma$

$$\begin{aligned}
\dot{\hat{\bar{\mu}}}(\sigma) &= \bar{\mu}(\chi_\sigma) = \overline{T'_\phi(\chi_\sigma)} = \overline{T_\phi(\chi_\sigma)} \\
&= \left(\sum_{\sigma' \in \Sigma} \phi(\sigma') \dot{\widehat{\chi_\sigma}}(\sigma') \right)^{-} \\
&= \left(\sum_{\sigma' \in \Sigma} \phi(\sigma') \int_G \chi_\sigma \overline{\chi_{\sigma^*}} \mathrm{d}\omega_G \right)^{-} \\
&= \left(\sum_{\sigma' \in \Sigma} \phi\,(\sigma')\, \delta_{\sigma',\sigma^*} \right)^{-} \\
&= \overline{\phi(\sigma^*)} = \phi(\sigma),
\end{aligned}$$

and the proof is complete. □

Remark 5.6. The boundedness condition added to the positive definiteness in the statement of Theorem 5.1 can be dropped once the underlying group G is Abelian. In the case $G := SU(2)$, however, one shows that the condition is necessary. In fact, let L be a non-continuous *-homomorphism on $F^0(G)$ and define a mapping ϕ on $\Sigma(G)$ by $\phi(\sigma) := L(\chi_\sigma)$ for all $\sigma \in \Sigma(G)$. Then ϕ is positive definite, but given $c > 0$, the non-continuity of L supplies a $\sigma \in \Sigma(G)$ satisfying $|L(\chi_\sigma)| > cd_\sigma$. Thus ϕ does not fulfill the boundedness condition.

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