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A CLASS OF ANTICIPATING LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we present the white noise methods for solving linear stochastic differential equations of anticipating type. Such equations may be solved using the S-transform, an important tool within the white noise theory. This approach provides a useful remedy to the fact that the Itô theory of stochastic integration is inapplicable to such equations. The technique is presented with several examples, including an application to finance.

1. Introduction

Let $B(t)$ be a Brownian motion and consider the stochastic integral equation

$$X(t) = X(a) + \int_a^t f(s, X(s))\, dB(s) + \int_a^t g(s, X(s))\, ds, \tag{1.1}$$

where $t \in [a,b], a, b \in [0, \infty)$. Equation (1.1) is an Itô stochastic integral equation if $X(a)$ is measurable with respect to $\sigma\{B(s) : s \le a\}$. This Itô stochastic integral equation has a unique continuous solution provided that f and g satisfy Lipschitz and growth conditions, that is

(i) there exists $C_1 > 0$ such that for any $t \in [a,b]$, and $x, y \in \mathbb{R}$,

$$|f(t,x) - f(t,y)| + |g(t,x) - g(t,y)| \le C_1|x-y|,$$

and

(ii) there exists C_2 such that for any $t \in [a,b]$, and $x \in \mathbb{R}$,

$$|f(t,x)|^2 + |g(t,x)|^2 \le C_2(1+x^2),$$

respectively.

We note that the existence of a solution is established by applying the Picard iteration, where $X_0(t) = X(a)$ and for $n \ge 1$,

$$X_n(t) = X(a) + \int_a^t f(s, X_{n-1}(s))dB(s) + \int_a^t g(s, X_{n-1}(s))\, ds$$

and with probability 1, $X_n(t)$ converges to $X(t)$ on $[a,b]$ uniformly. Interestingly enough, if we relax the measurability requirement on $X(a)$ (with respect to $\sigma\{B(s) : s \le a\}$), $f(s, X(a))$ may be anticipating. As a result $\int_a^t f(s, X(a))\, dB(s)$ is not an Itô integral, hence $X_1(t)$ is undefined as an Itô process. Moreover,

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Equation (1.1) would no longer be an Itô stochastic integral equation. Questions emanating from this situation have been considered by several researchers. In particular, we mention the works by Buckdahn [1][2], Léon and Protter [8] and Nualart and Pardoux [9], some of which employ techniques from the Malliavin Calculus. The notation and terminology used throughout this paper is standard as may be seen for instance in the books [4][6][10] .

2. The White Noise Methods

We give a brief review of the white noise theory from the book [6].

2.1. The white noise space. Let E be a separable Hilbert space with norm $|\cdot|_0$. Let A be a densely defined self-adjoint operator on E, whose eigenvalues $\{\lambda_n\}_{n\geq 1}$ satisfy the conditions

- $1 < \lambda_1 \leq \lambda_2 \leq \cdots,$
- $\sum_{n=1}^{\infty} \lambda_n^{-2} < \infty.$

For any $p \geq 0$, let $\mathcal{E}_p$ be the completion of E with respect to the norm $|f|_p = |A^p f|_0$. Observe that $\mathcal{E}_p$ is a Hilbert space under the norm $|\cdot|_p$, and $\mathcal{E}_p \subset \mathcal{E}_q$ for all $p \geq q$.

In fact, by the second condition on the eigenvalues of A, the inclusion map $i : \mathcal{E}_{p+1} \to \mathcal{E}_p$ is a Hilbert-Schmidt operator (see [6] for details).

Next, let $\mathcal{E}$ = projective limit of $\{\mathcal{E}_p : p \geq 0\}$ and let $\mathcal{E}'$ be its dual.The space $\mathcal{E} = \cap_{p\geq 0}\mathcal{E}_p$ equipped with the topology given by the family $\{|\cdot|_p\}_{p\geq 0}$ of semi-norms is a nuclear space. Consequently $\mathcal{E} \subset E \subset \mathcal{E}'$ is a Gel'fand triple with continuous inclusions:

$$\mathcal{E} \subset \mathcal{E}_q \subset \mathcal{E}_p \subset E \subset \mathcal{E}_p' \subset \mathcal{E}_q' \subset \mathcal{E}', \quad q \geq p \geq 0,$$

having identified E with itself using the Riesz Representation Theorem.

Let $\langle \cdot, \cdot \rangle$ denote the duality theorem between $\mathcal{E}'$ and $\mathcal{E}$. By Minlos theorem, there is a unique probability measure μ on the Borel subsets of $\mathcal{E}'$ such that for any $f \in \mathcal{E}$, the random variable $\langle \cdot, f\rangle$ is normally distributed with mean 0 and variance $|f|_0^2$. It follows that μ is uniquely determined by

$$\int_{\mathcal{E}'} e^{i\langle x,\xi\rangle}\, d\mu = e^{-\frac{1}{2}|\xi|_0^2}, \quad \forall \xi \in \mathcal{E}. \tag{2.1}$$

The probability space $(\mathcal{E}', \mu)$ is known as the *white noise space*. We denote the space $L^2(\mathcal{E}', \mu)$ by (L^2), observing that this space consists of all measurable functions $h : \mathcal{E}' \to \mathbb{C}$ such that

$$\int_{\mathcal{E}'} |h(x)|^2\, d\mu(x) < \infty.$$

Within this framework, the somewhat ubiquitous white noise tool, known as the S-transform is defined (see [6]). In fact, if $\varphi(t) \in (L^2)$, then for $\xi \in S_c$,

$$S\varphi(t)(\xi) = \int_{S'} \varphi(t)(x+\xi)\, d\mu(x).$$

Meantime, if X and Y are generalized functions, their Wick product, denoted $X \diamond Y$ is the unique generalized function such that

$$S(X \diamond Y) = (SX)(SY).$$

Clearly, an important feature of the S-transform is that much like the Fourier transform changes convulotions into products, it turns Wick products into ordinary products. It is worthnoting that the Wick product plays an intrinsic role in stochastic integration, especially when one discusses situations involving anticipating initial conditions or integrands.

2.2. The Hitsuda-Skorohod integral. Let $\partial_t \equiv D_{\partial_t}$ be the *white noise differential operator* (also known as the *Hida differential operator* or the *annihilation operator*), as defined in [6]. The adjoint of ∂_t, denoted by $\partial_t^* \equiv D_{\partial_t}^*$, is called the *creation operator*.

Starting with the Gel'fand triple $S(\mathbb{R}) \subset L^2(\mathbb{R}) \subset S'(\mathbb{R})$ and following [6], one obtains the Gel'fand triple $(S)_\beta \subset (\mathcal{L}^2) \subset (S)_\beta^*$. If $\varphi : [a,b] \to (S)_\beta^*$ is Pettis integrable, then the (white noise) integral $\int_a^b \partial_t^* \varphi(t)\, dt$ is called the *Hitsuda-Skorohod integral* of φ, provided $\int_a^b \partial_t^* \varphi(t)\, dt$ is a random variable in (L^2).

The Hitsuda-Skorohod integral extends the Itô integral to $\varphi(t)$ which may be anticipating. In fact, if $\varphi(t)$ is nonanticipating and $\int_a^b \| \varphi(t)\|_0^2\, dt < \infty$, then

$$\int_a^b \partial_t^* \varphi(t)\, dt = \int_a^b \varphi(t)\, dB(t).$$

See [5] or [6] for details.

2.3. The white noise approach. In a bid to circumvent the challenges mentioned in Section 1 above, one possibility is to replace Equation (1.1), with

$$X(t) = X(a) + \int_a^t \partial_s^* f(s, X(s))\, ds + \int_a^t g(s, X(s))\, ds, \tag{2.2}$$

where $\int_a^t \partial_s^* f(s, X(s))\, ds$ is a Hitsuda-Skorohod integral. Equality in (2.2) is as random variables in the complex Hilbert space $(\mathcal{L}^2) \equiv \mathcal{L}^2(\xi', \mu)$.

The white noise methods involve using the S-transform to convert Equation (2.2) into

$$SX(t)(\xi) = SX(a)(\xi) + \int_a^t \xi(s) Sf(s, X(s))(\xi)\, ds + \int_a^t Sg(s, X(s))(\xi)\, ds \tag{2.3}$$

which is an ordinary integral equation for each fixed $\xi \in \mathcal{S}_c$. Next assuming Equation (2.3) can be solved for each ξ, a solution to Equation (2.2) would be obtained by applying the inverse S-transform, provided of course that taking inverse S-transform is possible. This requires that the solution to Equation (2.3) be in the range of the S-transform of an appropriate space.

3. Some Examples

We now turn our attention to a few interesting examples previously considered in turn by Buckdahn [1][2] and Kuo [6], using different techniques. In order to explain the key ideas, we describe the arguments from [6] for both examples.

Example 3.1. [1][6, page 280] Let us examine

$$X(t) = \mathrm{sgn}(B(1)) + \int_0^t X(s)\, dB(s).$$

Since $\mathrm{sgn}(B(1)) \notin \sigma\{B(s); s \leq 1\}$, the preceding equation corresponds to

$$X(t) = \mathrm{sgn}(B(1)) + \int_0^t \partial_s^* X(s)\, ds, \quad t \in [0, 1]. \tag{3.1}$$

So Equation (3.1) is not an Itô stochastic integral equation.

To solve Equation (3.1), using the approach in section 2, let $SX(t) = F(t)$ and $S[\,\mathrm{sgn}(B(1))] = G$. Then Equation (3.1) becomes (after we take S-transform)

$$F(t)(\xi) = G(\xi) + \int_0^t \xi(s) F(s)(\xi)\, ds,$$

so that for each $\xi \in S_c$, we have, with $t \in [0, 1]$,

$$F'(t) = \xi(t)F(t) \quad \text{and} \quad F(0) = G(\xi).$$

Consequently

$$\begin{aligned} F(t)(\xi) &= G(\xi)\, e^{\int_0^t \xi(s)\, ds} \\ &= G(\xi)\, e^{\langle 1_{[0,t)}, \xi\rangle} \\ &= G(\xi)\, S\left(: e^{\langle \cdot, 1_{[0,t)}\rangle} :\right)(\xi) \\ &= S(\mathrm{sgn}(B(1)))(\xi)\, S\left(: e^{\langle \cdot, 1_{[0,t)}\rangle} :\right)(\xi) \\ &= S\left[\{\mathrm{sgn}(B(1))\} \diamond \left(: e^{\langle \cdot, 1_{[0,t)}\rangle} :\right)\right](\xi), \end{aligned} \tag{3.2}$$

whence we have

$$\begin{aligned} X(t) &= \{\mathrm{sgn}(B(1))\} \diamond \left(: e^{\langle \cdot, 1_{[0,t)}\rangle} :\right) \\ &= \{\mathrm{sgn}(B(1))\} \diamond \left(e^{B(t) - \frac{t}{2}}\right). \end{aligned}$$

It remains to show that $X(t) \in (L^2)$ for all t. To this end, consider

$$\varphi(t) = \mathrm{sgn}(B(1) - t)\, e^{B(t) - \frac{t}{2}} = \mathrm{sgn}(\langle \cdot, 1_{[0,1]}\rangle - t)\, e^{\langle \cdot, 1_{[0,t)}\rangle - \frac{t}{2}},$$

since $B(t) = \langle \cdot, 1_{[0,t)}\rangle$.

Next, let us determine $S\varphi(t)$. By definition,

$$\begin{aligned} S\varphi(t)(\xi) &= \int_{S'} \varphi(t)(x+\xi)\, d\mu(x) \\ &= \int_{S'} \operatorname{sgn}\left(\langle x+\xi, 1_{[0,1]}\rangle - t\right)\, e^{\langle x+\xi, 1_{[0,t)}\rangle - \frac{t}{2}}\, d\mu(x) \\ &= \int_{S'} \operatorname{sgn}\left(\langle y+\xi, 1_{[0,1]\rangle}\right)\, e^{\langle 1_{[0,t)}, \xi\rangle}\, d\mu(y) \\ &= e^{\langle 1_{[0,t)}, \xi\rangle}\, S\left(\operatorname{sgn}(B(1))(\xi)\right) \\ &= F(t)(\xi) \qquad \text{(by Equation 3.2)} \end{aligned}$$

Recall that $F(t) = SX(t)$, so we see here that $SX(t) = S\varphi(t)$, with $\varphi(t) \in (L^2)$ and since S is injective, it follows that

$$X(t) = \varphi(t) = \operatorname{sgn}(B(1)-t)\, e^{B(t)-\frac{t}{2}}.$$

Example 3.2. [6, page 282] Consider the stochastic integral equation

$$X(t) = 1 + \int_0^t \partial_s^* X(s)\, ds + \int_0^t \operatorname{sgn}(B(1)-s)\, e^{B(s)-\frac{s}{2}}\, ds \tag{3.3}$$

where $0 \le t \le 1$. We claim that the solution to Equation (3.3) is given by $X(t) = e^{B(t)-\frac{t}{2}} + t\varphi(t)$, where $\varphi(t) = \operatorname{sgn}(B(1)-t)\, e^{B(t)-\frac{t}{2}}$.

Let $F(t) = S(X(t))$ and $G = S[\operatorname{sgn}(B(1))]$. Then we recall from Example 3.1 that for

$$\begin{aligned} \varphi(t) &= \operatorname{sgn}(B(1)-t)\, e^{B(t)-\frac{t}{2}} \\ &= \operatorname{sgn}\left(\langle \cdot, 1_{[0,1]}\rangle - t\right)\, e^{\langle \cdot, 1_{[0,t)}\rangle - \frac{t}{2}}, \end{aligned}$$

we have $S\varphi(t)(\xi) = e^{\langle 1_{[0,t)}, \xi\rangle}\, S(\operatorname{sgn}(B(1)))(\xi)$. Therefore applying the S-transform to (3.3), we get for $\xi \in S_c$

$$S(X(t))(\xi) = 1 + \int_0^t \xi(s) S(X(s))(\xi)\, ds + \int_0^t S(\varphi(s))(\xi)\, ds$$

so that

$$\begin{aligned} F(t)(\xi) &= 1 + \int_0^t \xi(s) F(s)(\xi)\, ds + \int_0^t e^{\langle 1_{[0,s)}, \xi\rangle} S(\operatorname{sgn}(B(1))(\xi)\, ds \\ &= 1 + \int_0^t \xi(s) F(s)(\xi)\, ds + \int_0^t G(\xi) e^{\langle 1_{[0,s)}, \xi\rangle}\, ds, \end{aligned}$$

since $G = S\operatorname{sgn}(B(1))$. So

$$F(t)(\xi) = 1 + \int_0^t \xi(s) F(s)(\xi)\, ds + \int_0^t G(\xi) e^{\int_0^s \xi(u)\, du}\, ds. \tag{3.4}$$

Now for each $\xi \in S_c$, Equation (3.4) implies that $F(t)$ satisfies the ordinary differential equation

$$\begin{aligned} &F'(t) = \xi(t) F(t) + G(\xi) e^{\int_0^t \xi(s)\, ds}, \\ &F(0) = 1, \end{aligned} \tag{3.5}$$

for $t \in [0,1]$. We now seek a solution for Equation (3.5), which is just a first order linear ordinary differential equation. We have

$$F'(t) - \xi(t)F(t) = G(\xi)e^{\int_0^t \xi(s)\,ds}$$

with integrating factor $e^{-\int_0^t \xi(s)\,ds}$. Multiplying through by the integrating factor, Equation (3.5) becomes

$$e^{-\int_0^t \xi(s)\,ds}F'(t) - \xi(t)F(t)e^{-\int_0^t \xi(s)\,ds} = G(\xi),$$

which is

$$\frac{d}{dt}\left[F(t)e^{-\int_0^t \xi(s)\,ds}\right] = G(\xi).$$

Therefore

$$F(t)e^{-\int_0^t \xi(s)\,ds} = \int_0^t G(\xi)\,ds + K,$$

where K is a constant. Hence $F(t)$ is given by

$$F(t) = tG(\xi)e^{\int_0^t \xi(s)\,ds} + Ke^{\int_0^t \xi(s)\,ds},$$

and since $F(0) = 1$, we have $K = 1$. So

$$F(t) = e^{\int_0^t \xi(s)\,ds}\{1 + tG(\xi)\} \tag{3.6}$$

is the solution to Equations (3.5).

Next, we recall that

$$e^{\int_0^t \xi(s)\,ds} = e^{\langle 1_{[0,t)},\xi\rangle} = S\left(e^{B(t)-\frac{t}{2}}\right)(\xi).$$

Moreover

$$S(\varphi(t))(\xi) = S\left(\operatorname{sgn}(B(1)-t)e^{B(t)-\frac{t}{2}}\right)(\xi) = G(\xi)e^{\int_0^t \xi(s)\,ds},$$

which now allows us to rewrite equation (3.6) as

$$S(X(t))(\xi) = S\left(e^{B(t)-\frac{t}{2}}\right)(\xi) + t\,S(\varphi(t))(\xi).$$

It then follows that

$$\begin{aligned} X(t) &= e^{B(t)-\frac{t}{2}} + t\,\varphi(t) \\ &= e^{B(t)-\frac{t}{2}} + t\operatorname{sgn}(B(1)-t)e^{B(t)-\frac{t}{2}} \\ &= e^{B(t)-\frac{t}{2}}\{1 + t\operatorname{sgn}(B(1)-t)\}. \end{aligned}$$

The preceding examples provide the basis for our first result:

Theorem 3.1. *Let $X(t)$ be a stochastic process such that*

$$S(X(t))(\xi) = G(\xi)e^{\int_0^t \xi(s)\,ds},$$

where $\xi \in S_c$. Then the solution to the stochastic integral equation

$$Y(t) = 1 + \int_0^t \partial_s^* Y(s)\,ds + \int_0^t X(s)\,ds \tag{3.7}$$

for $t \in [0,1]$ is given by $Y(t) = e^{B(t)-\frac{t}{2}} + t\,X(t)$.

Proof. Let $SY(t) = H(t)$. By hypothesis $S(X(t)) = G(\xi)e^{\int_0^t \xi(s)\,ds}$. Applying the S-transform to Equation (3.7), we have

$$S(Y(t))(\xi) = 1 + \int_0^t \xi(s)S(Y(s))(\xi)\,ds + \int_0^t S(X(s))(\xi)\,ds \tag{3.8}$$

$$\Rightarrow \quad H(t)(\xi) = 1 + \int_0^t \xi(s)H(s)(\xi)\,ds + \int_0^t G(\xi)\,e^{\int_0^s \xi(u)\,du}ds$$

So for each $\xi \in S_c$, we have

$$\begin{aligned}
& H'(t) = \xi(t)H(t) + G(\xi)\,e^{\int_0^t \xi(u)\,du}, \quad H(0) = 1 \\
\Rightarrow \quad & \frac{d}{dt}\left[H(t)\,e^{-\int_0^t \xi(u)\,du}\right] = G(\xi) \\
\Rightarrow \quad & H(t)\,e^{-\int_0^t \xi(u)\,du} = t\,G(\xi) + K \\
\Rightarrow \quad & H(t) = t\,G(\xi)\,e^{\int_0^t \xi(u)\,du} + K\,e^{\int_0^t \xi(u)\,du}
\end{aligned}$$

Since $H(0) = 1$ and $K = 1$, we have

$$H(t) = e^{\int_0^t \xi(u)\,du} + t\,G(\xi)\,e^{\int_0^t \xi(u)\,du},$$

namely

$$S(Y(t))(\xi) = S\left(e^{B(t)-\frac{t}{2}}\right)(\xi) + t\,S(X(t))(\xi),$$

which implies $Y(t) = e^{B(t)-\frac{t}{2}} + t\,X(t)$, as desired. □

4. A Class of Linear Equations

We now consider a class of equations based on the general linear stochastic integral equation of Hitsuda-Skorohod type, namely

$$X(t) = \varphi + \int_a^t \partial_s^*(f(s)X(s))\,ds + \int_a^t [g(s)X(s) + \psi(s)]\,ds,$$

where f, g are deterministic, φ is a random variable and ψ is a stochastic process. We will see how the S-transform can be used to solve this equation. We start by recalling a lemma (Lemma 13.32 in [6]) and a theorem (Theorem 13.33 in [6]) and close with another unifying result.

Lemma 4.1. [6] *If $f \in \mathcal{L}^2([a,b])$ and $\varphi \in \mathcal{L}^p(S')$ for some $p > 2$, then*

$$\varphi \diamond e^{[\int_a^t f(s)\,dB(s) - \frac{1}{2}\int_a^t f(s)^2\,ds]} = \left(T_{-1_{[a,t]}f}\varphi\right)\,e^{[\int_a^t f(s)\,dB(s) - \frac{1}{2}\int_a^t f(s)^2\,ds]},$$

where $\diamond$ is the Wick product and $T_h\varphi(x) = \varphi(x+h)$.

Next we consider a result that shows the solution of the general Hitsuda-Skorohod type stochastic integral equation when certain conditions are specified.

Theorem 4.2. [6] *Suppose $f(t)$, $g(t)$ are deterministic functions, φ is a random variable, and $\psi(t)$ is a stochastic process satisfying*

1. *$f, g \in \mathcal{L}^2([a,b]))$.*
2. *$\varphi \in \mathcal{L}^p(S')$ for some $p > 2$.*
3. *$\psi \in \mathcal{L}^q([a,b] \times S')$ for some $q > 2$.*

Then the stochastic integral equation

$$X(t) = \varphi + \int_a^t \partial_s^*(f(s)X(s))\,ds + \int_a^t (g(s)X(s) + \psi(s))\,ds$$

has a unique solution in $\mathcal{L}^2([a,b],(L^2))$ *given by*

$$\begin{aligned} X(t) &= \left(T_{-1_{[a,t)}f}\varphi\right) e^{\left[\int_a^t f(s)\,dB(s) + \int_a^t \left(g(s) - \frac{1}{2}f(s)^2\right)ds\right]} \\ &\quad + \int_a^t \left(T_{-1_{[s,t)}f}\psi(s)\right) e^{\left[\int_s^t f(r)\,dB(r) + \int_s^t \left(g(r) - \frac{1}{2}f(r)^2\right)dr\right]}\,ds. \end{aligned}$$

Finally, for simplicity, let

$$\mathcal{E}_f(t) = e^{\left[\int_0^t f(s)\,dB(s) - \frac{1}{2}\int_0^t f(s)^2\,ds\right]}.$$

In view of the general linear Hitsuda-Skorohod type stochastic integral equation and the concluding result in the previous section, we have

Theorem 4.3. *If* $f \in \mathcal{L}^2([0,1])$ *and* φ *is a random variable, with* $\varphi \in \mathcal{L}^p(S')$ *for some* $p > 2$*. Then the stochastic integral equation*

$$Y(t) = 1 + \int_0^t \partial_s^*(f(s)Y(s))\,ds + \int_0^t (\varphi \diamond \mathcal{E}_f(s))\,ds \tag{4.1}$$

has a solution given by

$$Y(t) = \mathcal{E}_f(t) + t(\varphi \diamond \mathcal{E}_f(t)), \quad t \in [0,1].$$

Proof. For simplicity, let $X(t) = \varphi \diamond \mathcal{E}_f(t)$, and let $SY(t) = F(t)$, $S\varphi = G$. Since $f \in \mathcal{L}^2([0,1])$, we have

$$S(\mathcal{E}_f(t))(\xi) = e^{\int_0^t \xi(s)f(s)\,ds}.$$

Therefore, by taking the S-transform of Equation (4.1), we have for $\xi \in S_c$,

$$F(t)(\xi) = 1 + \int_0^t \xi(s)f(s)F(s)(\xi)ds + \int_0^t G(\xi)\,e^{\int_0^s \xi(u)f(u)\,du}\,ds,$$

so that for each fixed $\xi \in S_c$,

$$F'(t) = \xi(t)f(t)F(t)(\xi) + G(\xi)\,e^{\int_0^t \xi(u)f(u)\,du}, \quad \text{with } F(0) = 1.$$

For each fixed $\xi \in S_c$, we have

$$\frac{d}{dt}\left[e^{-\int_0^t \xi(s)f(s)\,ds}F(t)\right] = G(\xi)$$

which yields

$$e^{-\int_0^t \xi(u)f(u)\,du}F(t) = t\,G(\xi) + K,$$

where K is a constant. Hence we have

$$F(t) = K\,e^{\int_0^t \xi(u)f(u)\,du} + t\,G(\xi)\,e^{\int_0^t \xi(u)f(u)\,du}.$$

Since $F(0) = 1$, we get $K = 1$ and thus

$$F(t) = e^{\int_0^t \xi(s)f(s)\,ds} + t\,G(\xi)\,e^{\int_0^t \xi(s)f(s)\,ds}.$$

The conclusion follows once we take the inverse S-transform, that is

$$Y(t) = \mathcal{E}_f(t) + t(\varphi \diamond \mathcal{E}_f(t)).$$

□

8Application

One situation where anticipating initial conditions and integrands arise is in the pricing of bonds. In [11], the authors propose a framework for determining price dynamics of a bond $P(t,T)$ at time t, which matures at time T with a fixed expiration value $P(T,T) = 1$ almost surely. In effect, this involves considering a stochastic process driven by $B(t)$ which reaches a fixed value at a future time T almost surely.

The main idea involves considering a stochastic integral equation of the form

$$X_t = X_T - \int_t^T f(s)X(s)dB(s) - \int_t^T g(s)X(s)ds \tag{4.2}$$

where $f(s)$ is anticipating. As the authors point out, one must note that even if $f(s)$ and $g(s)$ are adapted, the terminal condition $X_T = 1$ makes for an anticipating equation. By introducing a time reversal operator (see [11]), Equation (4.2) becomes

$$X_t = X_0 - \int_0^t f(s)X(s)dB(s) - \int_0^t g(s)X(s)ds \tag{4.3}$$

with $x(0) = P(T,T) = x_0$.

Considering Equation (4.3), the authors assert the existence of a unique solution given by

$$x(t) = x_0\mathcal{E}_f(t)\text{exp }\{\int_0^t g(s)ds\}.$$

We will now derive this solution using the white noise methods discussed in this paper. Indeed, in view of Equation (4.2), let $SX(t) = F(t), G = Sx_0$.

Then upon taking S-transforms, we have

$$F(t)(\xi) = G(\xi) + \int_0^u [\xi(s)f(s) + g(s)]F(s)(\xi)ds$$

almost surely, for all $u \in [0,t]$. Using similar arguments as in Theorem 4.2, let

$$H_\xi(u) = G(\xi) + \int_0^u v(s)F(s)(\xi)ds$$

where $v(u) = \xi(u)f(u) + g(u)$, so that u-a.s., $H_\xi(u) = F(u)(\xi), \forall\xi \in S_C$, and $H'_\xi(u) = v(u)H_\xi(u)$, u-a.s. in $[0,t]$ with $H_\xi(0) = G(\xi)$. Therefore,

$$H'_\xi(u) - v(u)H_\xi(u) = 0$$

$$\frac{d}{du}\{H_\xi(u)e^{-\int_0^u v(s)ds}\} = 0.$$

Since $H_\xi(0) = G(\xi), R = G(\xi)$ and so $H_\xi(u) = G(\xi)e^{\int_0^u v(s)ds}$.

$$F(u)(\xi) = G(\xi)e^{\int_0^u v(s)ds},$$

u-a.e. on $[0,t]$. Next, since $f \in L^2([0,t])$,

$$S(\mathcal{E}_f(u)) = e^{\int_0^u \xi(s)f(s)ds}.$$

Consequently, we have

$$S(x_0\mathcal{E}_f(u) = (Sx_0)(\xi)[S\mathcal{E}_f(u)(\xi)] = G(\xi)e^{\int_0^u \xi(s)f(s)ds}.$$

$$F(u)(\xi) = G(\xi)e^{\int_0^u \xi(s)f(s)ds} \cdot e^{\int_0^u g(s)ds} = G(\xi)S\varepsilon_f(u)(\xi)e^{\int_0^u g(s)ds}.$$

It follows upon inverting that

$$X(t) = x_0\mathcal{E}_f(u)\exp\{\int_0^u g(s)ds\},$$

as desired.

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